ANALYSIS DIGEST

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This is a summary of basic undergraduate analysis, both real and complex. The application in mind is towards analytic number theory. I have attempted to actually give the proper definitions and at least sketch proofs of the important results. It would be a terrible idea to try and learn analysis from this, but hopefully it will serve as a reminder of things once learnt but forgotten. Comments and corrections welcome at `tb634@cam.ac.uk`.

1. Real Analysis

Given any $E \subset \mathbb{R}$ we define the outer measure as

$$\lambda^*(E) = \inf_{E \subset \cup (a_i, b_i)} \sum_{k=1}^{\infty} (b_i - a_i).$$

A set E is Lebesgue measurable if, for every $A \subset \mathbb{R}$,

$$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c).$$

We can then define its measure $\lambda(E) = \lambda^*(E)$ (so λ is countably additive on disjoint sets). The class of measurable sets is a σ-algebra, in that it is closed under complements and countable unions. Every interval is measurable.

A meausurable function is a function $f : \mathbb{R} \to \mathbb{R}$ such that $\{x : f(x) > \alpha\}$ is a measurable set for every $\alpha \in \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is simple if it only takes finitely many values, so that

$$f(x) = \sum_{i=1}^{n} a_i 1_{E_i}(x),$$

and without loss of generality we can assume the a_i are distinct and the E_i are disjoint and non-empty. If $f : \mathbb{R} \to [0, \infty)$ is a measurable simple function (i.e. each of the E_i is measurable) then by definition

$$\int f(x)\,\mathrm{d}x = \sum a_i \mu(E_i).$$

We extend this to all measurable functions $f : \mathbb{R} \to [0, \infty]$ by defining

$$\int f(x)\,\mathrm{d}x = \sup \left(\int_{\mathbb{R}} s(x)\,\mathrm{d}x \right)$$

where the supremum is over all $s : \mathbb{R} \to [0, \infty)$ which are measurable simple functions such that $s \leq f$. If E is a measurable set then

$$\int_E f(x)\,\mathrm{d}x = \int f(x) 1_E(x)\,\mathrm{d}x.$$

We can extend the definition of integral to functions $f : \mathbb{R} \to \mathbb{R}$ by defining

$$\int f(x)\,\mathrm{d}x = \int f^+(x)\,\mathrm{d}x - \int f^-(x)\,\mathrm{d}x,$$

where $f^+ = \max(f, 0)$ and $f^- = f^+ - f$. We can similarly extend to $f : \mathbb{R} \to \mathbb{C}$ by splitting up into real and imaginary parts. We say that f is integrable if $\int |f| < \infty$. It is easy to check from the definition that the integral is linear and monotone, so that if $f \leq g$ then $\int f \leq \int g$.

Theorem 1 (Monotone Convergence Theorem)**.** *If $f_n : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is a monotone increasing sequence of measurable non-negative functions then*

$$\int \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \int f_n(x)\,\mathrm{d}x.$$

Proof. For each j we have $f_j(x) \leq \lim_{n\to\infty} f_n(x) = f(x)$ and hence

$$\int f_j(x)\,\mathrm{d}x \leq \int f(x)\,\mathrm{d}x,$$

and the sequence of integrals $\int f_n$ is monotone increasing, so that

$$\lim_{n\to\infty} \int f_n = \sup \int f_n \leq \int f.$$

For the reverse inequality, let $s : \mathbb{R} \to \mathbb{R}$ be a simple measurable function such that $s \leq f$. Fix some $\alpha \in (0,1)$ and let $E_n = \{x \in \mathbb{R} : f_n(x) \geq \alpha s(x)\}$, so that $E_n \subset E_{n+1}$. Let $F_1 = E_1$, and for $n \geq 2$ let $F_n = E_n \backslash E_{n-1}$, so that $F_1, F_2, \ldots$ are measurable disjoint sets. Moreover, since each x is in some E_n for large enough n, we know that $\mathbb{R} = \cup F_i$. Countable additivity implies that

$$\begin{aligned}\alpha \int s(x)\,\mathrm{d}x &= \alpha \sum_i \int_{F_i} s(x)\,\mathrm{d}x \\ &= \alpha \lim_{n\to\infty} \sum_{1\leq i\leq n} \int_{F_i} s(x)\,\mathrm{d}x \\ &= \alpha \lim_{n\to\infty} \int_{E_n} s(x)\,\mathrm{d}x \\ &\leq \lim_{n\to\infty} \int f_n(x)\,\mathrm{d}x.\end{aligned}$$

Since $\alpha < 1$ was arbitrary, it follows that $\int s \leq \lim_{n\to\infty} f_n$, and so by definition $\int f \leq \lim_{n\to\infty} f_n$. □

Lemma 1 (Fatou's Lemma)**.** *If $f_n : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is a sequence of measurable non-negative functions then*

$$\int \liminf_{n\to\infty} f_n(x)\,\mathrm{d}x \leq \liminf_{n\to\infty} \int f_n(x)\,\mathrm{d}x.$$

Proof. Let $g_(x) = \inf_{k\geq n} f_k(x)$ so that $g_n \leq f_n$ and the sequence $g_n(x)$ is monotone increasing. By the monotone convergence theorem

$$\begin{aligned}\int \liminf_{n\to\infty} f_n(x)\,\mathrm{d}x &= \int \lim_{n\to\infty} g_n(x)\,\mathrm{d}x \\ &= \lim_{n\to\infty} \int g_n(x)\,\mathrm{d}x \\ &= \liminf_{n\to\infty} \int g_n(x)\,\mathrm{d}x \\ &\leq \liminf_{n\to\infty} \int f_n(x)\,\mathrm{d}x.\end{aligned}$$

□

Theorem 2. *If $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of measurable functions such that $f_n \to f$ pointwise and $|f_n(x)| \leq g(x)$ for some integrable function g and almost all x then*

$$\lim_{n\to\infty} \int f_n(x)\,\mathrm{d}x = \int \lim_{n\to\infty} f(x)\,\mathrm{d}x.$$

Proof. Let $f(x)$ be the pointwise limit of $f_n(x)$. We have that $|f(x)| \leq g(x)$. It follows that

$$\left| \int f - \int f_n \right| \leq \int |f - f_n|\,.$$

Since $|f - f_n| \leq 2g$, we may apply Fatou's lemma to the function $2g(x) - |f(x) - f_n(x)|$, so that

$$\int \liminf_{n\to\infty} 2g(x) - |f(x) - f_n(x)| \leq \liminf_{n\to\infty} \int 2g(x) - |f(x) - f_n(x)|\,.$$

Since g is integrable, we deduce that

$$\limsup_{n\to\infty} \int |f - f_n| \le \int \limsup_{n\to\infty} |f - f_n| = 0.$$

It follows that the limit exists and is 0. Hence

$$\lim_{n\to\infty} \left| \int f - \int f_n \right| \le \lim_{n\to\infty} \int_S |f - f_n| = 0.$$

□

Lemma 2 (Mean value theorem)**.** *If $f : [a,b] \to \mathbb{R}$ is continuous then there exists $c \in (a,b)$ such that*

$$\int_a^b f(x)\,\mathrm{d}x = f(c)(b-a).$$

If $f : [a,b] \to \mathbb{R}$ is continuous and differentiable on (a,b) then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

Proof. Let $m \le M$ be such that $f([a,b]) = [m,M]$. We have

$$m \le \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x \le M.$$

The result follows from the intermediate value theorem.

For the second, translating by rx if necessary for some r, we can assume that $f(a) = f(b)$. Again, let $f([a,b]) = [m,M]$. If both maximum and minimum are attained at a and b then f is constant so $f' = 0$. Suppose that the maximum M is attained at some $c \in (a,b)$. If we choose $h > 0$ small enough such that $c + h \in (a,b)$ then

$$\frac{f(c+h) - f(c)}{h} \le 0.$$

If $h < 0$ is small enough then

$$\frac{f(c+h) - f(c)}{h} \ge 0.$$

Since f is differentiable at c the limits as $h \to 0^-$ and $h \to 0^+$ must agree, so $f'(c) = 0$. □

Theorem 3 (Fundamental Theorem of Calculus)**.** *If $f : [a,b] \to \mathbb{R}$ is continuous then the function*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is uniformly continous on $[a,b]$ and is differentiable on (a,b), where $F'(x) = f(x)$. In particular, if $f : [a,b] \to \mathbb{R}$ is continuous and $F'(x) = f(x)$ for $x \in [a,b]$ then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

Proof. The first part follows from the mean-value theorem and the definition

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}.$$

The second part follows from the fact that if $f' = 0$ then f is constant, which also follows from the mean-value theorem. □

1.1. **Other notions of integral.** If $f : [a,b] \to \mathbb{R}$ is a bounded function then we say that it is Darboux integrable if

$$\sup \sum_i (x_i - x_{i-1}) \inf_{x_{i-1} \leq x \leq x_i} f(x) = \inf \sum_i (x_i - x_{i-1}) \sup_{x_{i-1} \leq x \leq x} f(x)$$

where the supremum and infimum are both taken over all partitions $a = x_0 < x_1 < \cdots < x_n = b$. This common value is defined to be $\int_a^b f(x)\,\mathrm{d}x$.

We say that $f : [a,b] \to \mathbb{R}$ is Riemann integrable if there is some I such that for all $\epsilon > 0$ there exists $\delta > 0$ such that for any partition $a = x_0 < x_1 < \cdots < x_n = b$ with $\max(x_i - x_{i-1}) < \delta$ and any choice of $t_i \in [x_{i-1}, x_i]$

$$\left| \sum (x_i - x_{i-1}) f(t_i) - I \right| < \epsilon.$$

We define $\int_a^b f(x)\,\mathrm{d}x = I$.

A bounded function on a compact interval is Riemann integrable if and only if it is Darboux integrable if and only if it is continuous at all points except possibly a set of Lebesgue measure zero.

If we change the definition of the Riemann integral to replace $(x_i - x_{i-1})$ by $g(x_i) - g(x_{i-1})$ for some $g : [a,b] \to \mathbb{R}$ then this is the Riemann-Stieltjes integral, denoted by

$$\int_a^b f(x)\,\mathrm{d}g(x).$$

This exists if, for example, f is continuous and g is of bounded variation on $[a,b]$, that is, $\sum |g(x_i) - g(x_{i-1})| = O(1)$ for every partition of $[a,b]$ (where the constant may depend on a, b, and g, but is independent of the partition). The Riemann-Stieltjes integral has the useful property of integration by parts, in that

$$\int_a^b f(x)\,\mathrm{d}g(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(x)\,\mathrm{d}f(x),$$

and either integral exists if and only if the other does. Furthermore, if g is continuously differentiable then

$$\int_a^b f(x)\,\mathrm{d}g(x) = \int_a^b f(x)g'(x)\,\mathrm{d}x$$

whenever the integral on the left-hand side exists.

2. Complex Analysis

As is traditional in analytic number theory we will often write $s = \sigma + it \in \mathbb{C}$ for an arbitrary complex variable, in which case $\sigma = \Re s$ is the real part of s and $t = \Im s$ is the imaginary part. The derivative of a function $f : \mathbb{C} \to \mathbb{C}$ at a point s is defined to be

$$f'(s) = \lim_{z \to s} \frac{f(z) - f(s)}{z - s}.$$

Implicit in this definition is the fact the limit exists and remains the same for any sequence $z_n \to s$. A neighbourhood of s is a bounded open set which contains s. We say that f is holomorphic on an open set U if $f'(s)$ exists for every $s \in U$, and that f is holomorphic at s if f is holomorphic on some neighbourhood of s. A function is entire if it is holomorphic on $\mathbb{C}$.

A smooth curve is a continuous function $\gamma : [a,b] \to \mathbb{C}$ with a non-vanishing continuous derivative which is injective (except possibly at the endpoints). More generally, a contour is a finite sequence of smooth curves joined at the endpoints. The contour integral of f along a smooth curve γ is

$$\int_\gamma f(s)\,\mathrm{d}s = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t,$$

which is extended in the obvious fashion for general contours.

Theorem 4 (Fundamental Theorem of Calculus)**.** *If $\gamma : [a,b] \to U$ where U is some open set, and if f is continuous at each point of γ, and $f = F'$ for some $F : U \to \mathbb{C}$ then*

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

Proof. By definition and the chain rule

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b F'(\gamma(t))\gamma'(t)\,\mathrm{d}t = \int_a^b (F\circ\gamma)'(t)\,\mathrm{d}t.$$

The result now follows by considering real and imaginary parts and applying the real fundamental theorem of calculus. □

Theorem 5 (Cauchy's theorem)**.** *If U is an open simply connected set, f is holomorphic on U, and γ is a closed contour in U, then*

$$\int_\gamma f(s)\,\mathrm{d}s = 0.$$

Proof. First note that it is true when f is a polynomial by the fundamental theorem of calculus. Now near z_0 approximate $f(z)$ by $f(z_0)+(z-z_0)f'(z_0)$. This proves the result if the diameter of γ is sufficiently small. To recover the result for arbitrarily large triangle contours subdivide the triangle into smaller triangles.

We can then deduce that if U is a convex region then, for any $a\in U$, the function $F(z)=\int_{[a,z]} f(w)\,\mathrm{d}w$ is holomorphic in U with $F'=f$, and hence by the fundamental theorem of calculus again this establishes the result for a convex region.

We can then recover the result for arbitrary polygonal contours by triangulating. Finally, approximate an arbitrary contour with a polygon (such that the area between the polygonal approximation and the actual contour is convex, so we can apply the previous case to 'move' the contour to the polygon). □

Theorem 6 (Cauchy integral formula)**.** *If D is a closed disc with boundary circle C and f is holomorphic on a neighbourhood of D then for every a in the interior of D*

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s-a}\,\mathrm{d}s.$$

Proof. By Cauchy's theorem the value of the integral remains unchanged if we move D to become a disc of radius r with a as the centre. Note that $\int_C \frac{1}{s-a}\,\mathrm{d}s = 2\pi i$. We then bound the difference

$$\left|\int_C \frac{f(s)}{s-a}\,\mathrm{d}s - 2\pi i f(a)\right| = \left|\int_0^{2\pi} \frac{f(a+re^{i\theta})-f(a)}{re^{i\theta}} re^{i\theta}\,\mathrm{d}\theta\right| \ll \sup_{\theta\in[0,2\pi]} \left|f(a+re^{i\theta})-f(a)\right|.$$

The right-hand side tends to 0 as $r\to 0$, since f is continuous at a. □

Theorem 7. *If f is holomorphic on some open set U then f has derivatives of all orders in U. Furthermore, for all $a\in U$ and $n\geq 1$,*

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_\gamma \frac{f(s)}{(s-a)^{n+1}}\,\mathrm{d}s,$$

where $\gamma\subset U$ is some circular contour containing a.

Proof. This follows from Cauchy's integral formula combined with the definition of derivative (and induction). □

Theorem 8. *If γ is a path and $f_1, f_2, \ldots$ are continous on γ such that there exist constants M_n with $\sum M_n$ converging and $|f_n(z)|\leq M_n$ for all n and $z\in\gamma$ then $f(z)=\sum_{n=1}^\infty f_n(z)$ exists, is continuous, and*

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{n=1}^\infty \int_\gamma f_n(z)\,\mathrm{d}z.$$

Theorem 9. *Every holomorphic function is analytic. That is, if f is holomorphic on some neighbourhood of a then there is some open disc centred at a in which f can be expanded as a convergent power series*

$$f(s) = \sum_{n=0}^\infty c_n(s-a)^n.$$

The coefficients c_n are

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-a)^{n+1}}\,\mathrm{d}w,$$

where C is any circle centred at a on and within which f is holomorphic.

Proof. Let γ be some small disc centred at a, so that for all z in this disc

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{s-z}\,\mathrm{d}s.$$

Expand out

$$\frac{1}{s-z} = \frac{1}{s-a}\left(\sum_{n=0}^{\infty}\frac{(z-a)^n}{(s-a)^n}\right)$$

and interchange the sum and integral, which is valid since

$$\left|\frac{(z-a)^n}{(s-a)^{n+1}f(s)}\right| \ll \frac{|z-a|^n}{r^{n+1}},$$

the sum of which converges since $|z-a|<r$. □

Theorem 10 (Identity Theorem). *If f and g are both holomorphic on an open and connected set D and $f=g$ for all $s\in S\subset D$, where S is such that there is some $x\in D$ such that every neighbourhood of x in D contains some point in S, then $f\equiv g$ on D.*

Theorem 11 (Maximum modulus principle). *If U is a connected open set and f is holomorphic on U, and if there exists some $a\in U$ such that $|f(a)|\geq|f(s)|$ for all s in a neighbourhood of a, then f is constant on U.*

Proof. By the identity theorem it suffices to show that if f is holomorphic on some disc D with centre a and $|f(z)|\leq|f(a)|$ for all $z\in D$ then f is constant on D. This follows from Cauchy's integral formula, since for any circular $\gamma\subset D$,

$$f(a) = \frac{1}{2\pi i}\int_\gamma\frac{f(z)}{z-a}\,\mathrm{d}z = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\,\mathrm{d}\theta,$$

and hence $|f(a)| = \frac{1}{2\pi}\int_0^{2\pi}\left|f(a+re^{i\theta})\right|\,\mathrm{d}\theta$. Therefore

$$\int_0^{2\pi}|f(a)| - \left|f(a+re^{i\theta})\right|\,\mathrm{d}\theta = 0$$

and so, since the integrand is continuous and non-negative, it must vanish everywhere, so $|f|$ is constant on D, which forces f to be constant. □

A function f is meromorphic at a point a if there is some neighbourhood of a on which either f or $1/f$ is holomorphic. In this case there is some n such that $(s-a)^n f(s)$ is holomorphic and non-zero in a neighbourhood of a. If $n>0$ then a is a pole of f of order n. If $n<0$ then a is a zero of f of order $-n$. If f is meromorphic at a then there is some neighbourhood of a in which f can be expressed as a Laurent series,

$$\sum_{n=-k}^{\infty} c_n(z-a)^n,$$

for some finite integer k. The coefficient c_{-1} is the residue of f at a, and is denoted by $\mathrm{Res}(f,a)$.

Theorem 12 (Residue theorem). *If U is a simply connected open set which contains a simple closed curve γ, f is holomorphic on γ, and is holomorphic inside γ except for a finite sequence $a_1,\ldots,a_k$, then*

$$\int_\gamma f(s)\,\mathrm{d}s = 2\pi i\sum_{i=1}^{k}\mathrm{Res}(f,a_k).$$

If U is a simply connected open set and f is holomorphic and non-zero on U then we define $\log f(z)$ on U as

$$\log f(z) = a + \int_b^z \frac{f'(s)}{f(s)}\,\mathrm{d}s,$$

where $b\in U$ and a is such that $\exp(a)=f(b)$. The integral can be taken over any path between b and z. This function is well-defined up to a constant (depending on the choice of a and b) which is always an integral multiple of $2\pi i$.