Analytic Number Theory Sheet 4 - Solutions

Lent Term 2020

1. Show that for $\sigma > 1/2$

$$\int_0^T |\zeta(\sigma+it)|^4 \, \mathrm{d}t \sim \frac{\zeta(2\sigma)^4}{\zeta(4\sigma)}T.$$

Solution: This question was mistakenly much more difficult than intended, since it suffers from the same defect as the proof of Theorem 15 in the lecture notes. I have below given a direct proof for those interested, and the proof of Theorem 15 can be fixed in a similar fashion.

We will use the approximate functional equation, with $x = y = (t/2\pi)^{1/2}$, which implies that for any $t \in [0, T]$,

$$\zeta(\sigma+it) = \sum_{n\leq x} \frac{1}{n^{\sigma+it}} + \chi(s)\sum_{n\leq x}\frac{1}{n^{1-\sigma-it}} + O(t^{-1/4}) = Z_1 + Z_2 + O(t^{-1/4}),$$

say. We will first show that

$$\int_0^T |Z_1|^4 \, \mathrm{d}t \sim \frac{\zeta(2\sigma)^4}{\zeta(4\sigma)}T.$$

Expanding out the fourth power and changing the order of summation and integral, the left-hand side is

$$\sum_{a,b,c,d\leq (T/2\pi)^{1/2}} (abcd)^{-\sigma}\int_M^T (ab/cd)^{-it}\,\mathrm{d}t.$$

where $M = 2\pi\max(a^2,b^2,c^2,d^2)$. The diagonal contribution where $ab = cd$ contributes

$$\sum_{a,b\leq (T/2\pi)^{1/2}} (ab)^{-2\sigma} \sum_{c,d\leq (T/2\pi)^{1/2}} 1_{ab=cd}(T-M).$$

Note that

$$\sum_{a,b,c,d\leq (T/2\pi)^{1/2}} 1_{ab=cd}(ab)^{-2\sigma} = \sum_{n\leq (T/2\pi)^{1/2}} \frac{\tau(n)^2}{n^{2\sigma}} + O\left(\sum_{(T/2\pi)^{1/2}<n\leq T/2\pi} \frac{\tau(n)^2}{n^{2\sigma}}\right).$$

Furthermore, using $\tau(n) \ll_\epsilon n^\epsilon$,

$$\sum_{n\leq x}\frac{\tau(n)^2}{n^{2\sigma}} = \sum_{n=1}^\infty \frac{\tau(n)^2}{n^{2\sigma}} + O_\epsilon(x^{1-2\sigma+\epsilon}) = \frac{\zeta(2\sigma)^4}{\zeta(4\sigma)} + O_\epsilon(x^{1-2\sigma+\epsilon})$$

and so

$$\sum_{a,b,c,d\leq (T/2\pi)^{1/2}} 1_{ab=cd}(ab)^{-2\sigma} = \frac{\zeta(2\sigma)^4}{\zeta(4\sigma)} + O_\epsilon(T^{1/2-\sigma+\epsilon})$$

Furthermore,

$$\sum_{a,b,c,d\leq (T/2\pi)^{1/2}} 1_{ab=cd}\frac{M}{(ab)^{2\sigma}} \ll \sum_{a,b,c,d\leq (T/2\pi)^{1/2}} 1_{ab=cd}\frac{a^2}{(abcd)^\sigma}$$

which is

$$\ll \sum_{a,b\leq (T/2\pi)^{1/2}} \frac{a^2\tau(ab)}{(ab)^{2\sigma}} \ll T^{\epsilon}(1+T^{3/2-\sigma}).$$

Altogether, then, the diagonal contribution is

$$\frac{\zeta(2\sigma)^4}{\zeta(4\sigma)}T + O_{\epsilon}(T^{3/2-\sigma+\epsilon}+T^{\epsilon}).$$

The non-diagonal contribution is

$$\ll \sum_{a,b,c,d\leq (T/2\pi)^{1/2}} (abcd)^{-\sigma}\frac{1}{\log(ab/cd)} \ll \sum_{n<m\leq T/2\pi} \frac{\tau(n)\tau(m)}{(mn)^{\sigma}\log(n/m)}.$$

Again using $\tau(n)\ll n^{\epsilon}$ this is

$$\ll_{\epsilon} T^{\epsilon} \sum_{n<m\leq T/2\pi} \frac{1}{(mn)^{\sigma}\log(m/n)}.$$

Using the familiar trick of bounding $\log(m/n) = -\log(1-\frac{m-n}{m}) > \frac{m-n}{m}$ this is $O_{\epsilon}(T^{2-2\sigma_{\epsilon}})$. Altogether then we have shown that

$$\int_0^T \left| \sum_{n\leq (T/2\pi)^{1/2}} n^{-\sigma-it}\right|^4 \mathrm{d}t \sim \frac{\zeta(2\sigma)^4}{\zeta(4\sigma)}T.$$

The contribution from the integral over Z_2 and the error term may, by similar calculations, be shown to be $o(T)$, and hence the claim follows.

2.

(a) Show that if F is a smooth function on $[0,1]$ then

$$|F(1/2)| \leq \int_0^1 (|F(t)| + \tfrac{1}{2}|F'(t)|)\,\mathrm{d}t.$$

(b) Let $t_1,\ldots,t_R \in [1/2, T-1/2]$ be a set of points such that whenever $i\neq j$ we have $|t_i - t_j|\geq 1$. Show that for any smooth $F:[0,T]\to\mathbb{C}$ we have

$$\sum_{1\leq i\leq R} |F(t_r)|^2 \leq \int_0^T \left(|F(t)|^2 + |F(t)F'(t)|\right)\,\mathrm{d}t.$$

(c) Deduce that for any $a_n\in\mathbb{C}$ we have

$$\sum_{1\leq i\leq R}\left|\sum_{n\leq N} a_n n^{it_r}\right|^2 \ll (T+N)\log N \sum_{n\leq N} |a_n|^2.$$

Solution: For part (a) we apply integration by parts to see that

$$\int_0^{1/2} F(t)\,\mathrm{d}t = \tfrac{1}{2}F(1/2) - \int_0^{1/2} tF'(t)\,\mathrm{d}t.$$

On the other hand, it also gives that

$$\int_{1/2}^1 F(t)\,\mathrm{d}t = \tfrac{1}{2}F(1/2) - \int_0^{1/2} (t-1)F'(t)\,\mathrm{d}t.$$

Summing both equalities shows

$$F(1/2) = \int_0^1 F(t)\,\mathrm{d}t + \int_0^{1/2} tF'(t)\,\mathrm{d}t + \int_{1/2}^1 (t-1)F'(t)\,\mathrm{d}t.$$

The bound in (a) follows by the triangle inequality.

For (b), we apply part (a) to the function $f(t) = F(t + t_r - 1/2)^2$ to see that

$$|F(t_r)|^2 \leq \int_{t_r-1/2}^{t_r+1/2} \left(|F(t)|^2 + |F(t)F'(t)|\right)\,\mathrm{d}t.$$

The result now follows summing over $1 \leq r \leq R$.

Finally, for (c), we apply the result in part (b) to the function $F(t) = \sum_{n\leq N} a_n n^{it}$. The mean-value estimate proved in lectures shows that

$$\int_0^T |F(t)|^2\,\mathrm{d}t \ll (T+N)\sum_{n\leq N} |a_n|^2.$$

Furthermore, by the Cauchy-Schwarz inequality,

$$\int_0^T |F(t)F'(t)|\,\mathrm{d}t \leq \left(\int_0^T |F(t)|^2\,\mathrm{d}t\right)^{1/2}\left(\int_0^T |F'(t)|^2\,\mathrm{d}t\right)^{1/2}.$$

The first factor we have already bounded. For the second, note that

$$F'(t) = \sum_{n\leq N} (a_n \log n) n^{it},$$

and so we can also use the mean-value estimate from lectures to see that

$$\int_0^T |F'(t)|^2\,\mathrm{d}t \ll (T+N)\sum_{n\leq N} |a_n \log n|^2 \ll (\log N)^2 (T+N)\sum_{n\leq N} |a_n|^2.$$

Combining these estimates with the upper bound from (b) gives the result.

3. By adapting the proof of Ingham given in lectures, show that if $c > 0$ is a constant such that $\zeta(\frac{1}{2}+iT) \ll T^c$ for all $T \geq 2$ then

$$N(\sigma, T) \ll T^{(2+4c)(1-\sigma)}(\log T)^{O(1)}$$

uniformly for $1/2 \leq \sigma \leq 1$. In particular, the Lindelöf hypothesis (that $\zeta(\frac{1}{2}+iT) \ll_\epsilon T^\epsilon$ for all $\epsilon > 0$) implies the Density Conjecture.

Solution: The main difference to the proof of Ingham as given in lectures comes when we need to bound

$$\int_0^T |\zeta(1/2+it)M(1/2+it)|^2\,\mathrm{d}t.$$

In the lectures we bounded this by using the Cauchy-Schwarz inequality and our upper bound on the 4th moment of $\zeta(1/2+it)$. If we have the pointwise bound $|\zeta(1/2+it)| \ll t^c$ available, however, then instead we can bound it above by

$$T^{2c}\int_0^T |M(1/2+it)|^2\,\mathrm{d}t \ll T^{1+2c}\log T$$

(recalling that $X \leq T$). Interpolating between the lines $\sigma = 1/2$ and $\sigma = 1+\delta$ then we arrive at the bound (using the notation from Ingham's proof)

$$\int_0^T |f_2(\sigma+it)|^2\,\mathrm{d}t \ll T^{(1+2c)(2-2\sigma)}(\log T)^{O(1)}.$$

Altogether, then, we now have an upper bound of

$$N(\sigma,T) \ll \left(TX^{1-2\sigma} + T^{(1+2c)(2-2\sigma)}\right)(\log T)^{O(1)}.$$

Making the simple choice of $X = T$ shows that the first summand is $\ll T^{2-2\sigma}$, and the claimed bound follows.

4. In this question we sketch an alternative approach to obtaining zero density estimates. Let $M(s) = \sum_{n\leq X}\frac{\mu(n)}{n^s}$, and for $1/2 \leq \alpha \leq 1$ let

$$R(\alpha) = \{\sigma + it : \alpha \leq \sigma \leq 1 \text{ and } T < t \leq 2T\}.$$

(a) Show that if $a_n = \sum_{d|n}\mu(d)1_{n/d\leq T}1_{d\leq X}$ then for all $s \in R(\alpha)$

$$\zeta(s)M(s) = \sum_{n\leq TX}\frac{a_n}{n^s} + O(T^{-\alpha}X^{1-\alpha}\log X).$$

(b) Show that if we choose $X^{1-\alpha} \leq T^{\alpha}(\log T)^{-2}$ and $X \leq T$ then, if $s \in R(\alpha)$ is a zero of $\zeta(s)$, for some $X \leq N \leq TX$ we have

$$\left|\sum_{N<n\leq 2N}\frac{a_n}{n^s}\right| \gg \frac{1}{\log T}.$$

(c) After making a suitable choice of X, combine the result of part (b) with the mean-value estimate of question 1 to deduce the zero density estimate

$$N(\alpha,T) \ll T^{4\alpha(1-\alpha)}(\log T)^{O(1)}.$$

Solution: Note that, by partial summation, in the region $R(\alpha)$

$$\zeta(s) = \sum_{n\leq T}\frac{1}{n^s} + O(T^{-\alpha}).$$

Using the trivial upper bound $M(s) \ll X^{1-\alpha}\log X$ therefore implies that

$$\zeta(s)M(s) = \left(\sum_{n\leq T}\frac{1}{n^s}\right)\left(\sum_{m\leq X}\frac{\mu(m)}{m^s}\right) + O(T^{-\alpha}X^{1-\alpha}\log X).$$

Part (a) now follows immediately, since the first term is just the product of two finite series, where the coefficient of k^{-s} is $\sum_{nm=k}1_{n\leq T}1_{m\leq X}\mu(m) = a_k$.

Suppose now that $s \in R(\alpha)$ is a zero of $\zeta(s)$. By part (a) it follows that

$$0 = 1 + \sum_{1<n\leq TX}\frac{a_n}{n^s} + O\left(T^{-\alpha}X^{1-\alpha}\log X\right).$$

By our choice of X the error term here is $O(1/\log T)$, and hence in particular at most $1/2$ for large enough T.

Now note that for $1 < n \leq X$ the coefficient a_n simplifies to just $\sum_{d|n}\mu(d)$, if we also assume that $X \leq T$, which is zero. Therefore we see that

$$\left|\sum_{X\leq n\leq TX}\frac{a_n}{n^s}\right| \geq \frac{1}{2}.$$

The claim in (b) now follows if we partition the left-hand side into $O(\log T)$ intervals of the shape $[N, 2N]$ and use the pigeonhole principle.

Suppose that we have K many zeros of $\zeta(s)$ in the region $R(\alpha)$. We know that the result in part (b) is true for some N for each zero. If we simultaneously pigeonhole, we can deduce that there exists some $X \leq N \leq TX$ that works for at least $\gg K/\log T$ many such zeros. Therefore, by Question 2, if we choose any R such zeros which are well-spaced then

$$R(\log T)^{-2} \ll (T+N)\log N \sum_{n\leq N} \frac{|a_n|^2}{n^{2\alpha}}.$$

Using the upper bound $|a_n| \leq \tau(n)$ we see that the sum here is $\ll N^{1-2\alpha}(\log N)^{O(1)}$. Therefore we have, recalling that $X \leq N \leq XT$,

$$R \ll (TX^{1-2\alpha} + (TX)^{2-2\alpha})(\log T)^{O(1)}.$$

Thus, choosing $X = T^{2\alpha-1}/(\log T)^2$, we have

$$R \ll T^{4\alpha(1-\alpha)}(\log T)^{O(1)}.$$

Note that that this choice of X satisfies the conditions in part (b) since $\alpha \in [1/2, 1]$ and $(1-\alpha)(2\alpha-1) = 3\alpha - 1 - 2\alpha^2 \leq \alpha$. The claimed upper bound on $N(\sigma, T)$ now follows since we can partition the full set of zeros into well-spaced sets at a cost of $O(\log T)$.

5. Fix some $\sigma > 1$.

(a) Show that for all t

$$|\zeta(\sigma + it)| \leq \zeta(\sigma).$$

(b) Show that for any $N \geq 1$ and $t \geq 0$

$$|\zeta(\sigma + it)| \geq \sum_{n=1}^{N} \frac{\cos(t\log n)}{n^\sigma} - \sum_{n>N} \frac{1}{n^\sigma}.$$

(c) Show that, for any $a_1, \ldots, a_N \in \mathbb{R}$ and $\epsilon > 0$ there exist arbitrarily large t such that there exist $m_1, \ldots, m_N \in \mathbb{N}$ with

$$|ta_n - m_n| \leq \epsilon$$

for $1 \leq n \leq N$.

(d) Show that, for any $\epsilon > 0$, there are arbitrarily large t such that

$$|\zeta(\sigma + it)| \geq (1-\epsilon)\zeta(\sigma).$$

Solution: Part (a) follows immediately from the existence of a Dirchlet series and the triangle inequality:

$$|\zeta(\sigma + it)| = \left|\sum_{n=1}^{\infty} \frac{1}{n^{\sigma+it}}\right| \leq \sum_{n=1}^{\infty} \frac{1}{n^\sigma} = \zeta(\sigma).$$

For part (b), we in fact give a lower bound for the real part of $\zeta(\sigma + it)$, by dividing into two sums:

$$\Re\zeta(\sigma + it) = \Re\sum_{n=1}^{N} \frac{1}{n^{\sigma+it}} + \Re\sum_{n>N} \frac{1}{n^{\sigma+it}}.$$

The second sum is at most $\sum_{n>N} \frac{1}{n^\sigma}$ in absolute value by the triangle inequality, and hence

$$\Re\zeta(\sigma + it) \geq \sum_{n=1}^{N} \Re\frac{1}{n^{\sigma+it}} - \sum_{n>N} \frac{1}{n^\sigma},$$

and part (b) follows.

Part (c) is a simple application of the pigeonhole principle - for any $(x_1, \ldots, x_N) \in \mathbb{R}^N$, consider the location of the fractional part vector $(\{x_1\}, \ldots, \{x_N\}) \in [0,1)^N$. If we divide this into at most $2^N \epsilon^{-N}$ boxes, each of width $\epsilon/2$ in every direction, then by the pigeonhole principle in any interval of length at least $4^N \epsilon^{-N}$, say, there exist two distinct t_1, t_2 such that the fractional parts of both $t_1 \cdot \mathbf{a}$ and $t_2 \cdot \mathbf{a}$ lie in the same box, and therefore the fractional parts of $(t_1 - t_2) \cdot \mathbf{a}$ are all in $[-\epsilon, \epsilon]$.

This shows the existence of at least one such $t \in [1, (4/\epsilon)^N]$. The existence of infinitely many such t follows by applying this same result to $T^k \cdot \mathbf{a}$ for some large T and all $k \in \mathbb{N}$.

Finally, for part (d), we apply the result of part (c) with $a_n = \frac{1}{\pi} \log n$, to find arbitrarily large t such that, for all $1 \leq n \leq N$,

$$|t \log n - \pi m_n| \leq \epsilon/2$$

for some integer m_n. Since cos has period π and $\cos(x) \geq 1 - |x|$ for $x \in [0, 1/4]$, it follows from part (b) that

$$\begin{aligned} |\zeta(\sigma + it)| &\geq \sum_{n=1}^{N} \frac{1 - \epsilon/2}{n^\sigma} - \sum_{n>N} \frac{1}{n^\sigma} \\ &= (1 - \epsilon/2)\zeta(\sigma) - (2 - \epsilon) \sum_{n>N} \frac{1}{n^\sigma}. \end{aligned}$$

The result follows from the fact that $\zeta(\sigma) > 1/(\sigma - 1)$ and $\sum_{n>N} \frac{1}{n^\sigma} < N^{1-\sigma}/\sigma - 1$ (if we choose N such that $4N^{1-\sigma} < \epsilon$).