Analytic Number Theory Sheet 3

Lent Term 2020

1. Using the fact that $\zeta(s)$ has no zeros in the region $\sigma > 1 - c/\log|t|$ and $|t| \geq 2$ prove that, all in this same region,

(a)

$$\frac{\zeta'}{\zeta}(s) \ll \log|t|$$

Hint: Use Dirichlet series to handle $\sigma > 1 + 1/\log|t|$ and apply the formula for $\frac{\zeta'}{\zeta}$ in terms of the zeros of $\zeta(s)$ to handle the remaining region.

(b)

$$|\log\zeta(s)| \leq \log\log|t| + O(1),$$

(c)

$$\frac{1}{\zeta(s)} \ll \log|t|.$$

Solution: When $\sigma > 1$,

$$\left|\frac{\zeta'}{\zeta}(s)\right| \leq \sum_n \frac{\Lambda(n)}{n^\sigma} \ll \frac{1}{\sigma - 1},$$

which provides the required estimate when $\sigma \geq 1 + 1/\log|t|$. Otherwise we use the fact that

$$\frac{\zeta'}{\zeta}(s) = \sum_\rho \frac{1}{s-\rho} + O(\log|t|)$$

uniformly in the region $5/6 \leq \sigma \leq 2$ and $|t| \geq 2$, where the sum is over all zeros ρ such that $|\rho - (3/2 + it)| \leq 5/6$.

Let $s = \sigma + it$ where $1 - c/2\log|t| < \sigma \leq 1 + 1/\log|t|$. Taking the difference of the above equation

$$\frac{\zeta'}{\zeta}(s) = \frac{\zeta'}{\zeta}(s_1) + \sum_\rho \left(\frac{1}{s-\rho} - \frac{1}{s_1-\rho}\right) + O(\log|t|),$$

where $s_1 = 1 + 1/\log|t| + it$. By the first part of the solution the first term is $O(\log|t|)$. By the zero-free region all of the zeros in the sum are a reasonable distance away from s, so in particular, $|s-\rho| \asymp |s_1-\rho|$ for all such ρ. Therefore

$$\frac{1}{s-\rho} - \frac{1}{s_1-\rho} \ll \frac{1}{|s_1-\rho|^2\log|t|} \ll \Re\frac{1}{s_1-\rho}.$$

By the formula again

$$\frac{\zeta'}{\zeta}(s_1) = \sum_\rho \frac{1}{s_1-\rho} + O(\log|t|)$$

and hence

$$\sum_\rho \Re\frac{1}{s_1-\rho} \ll \log|t|$$

and we are done.

For part (b), in the region $\sigma \geq 1 + 1/\log|t|$, by the Dirichlet series representation we get the estimate

$$|\log \zeta(s)| \ll \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-\sigma} = \log \zeta(\sigma) < \log(1 + 1/(\sigma - 1)) \leq \log\log|t| + O(1).$$

Again, for the region $1 - c/2\log|t| < \sigma \leq 1 + 1/\log|t|$ we consider $s_1 = 1 + 1/\log|t| + it$ and the difference $\log \zeta(s) - \log \zeta(s_1)$ to extend the estimate for s_1 down to s. In this case we use the identity that

$$\log \zeta(s) - \log \zeta(s_1) = \int_{s_1}^{s} \frac{\zeta'}{\zeta}(z)\,\mathrm{d}z,$$

which is valid since there are no zeros of $\zeta(s)$ on the line. By part (a) this difference is $O(1)$ and we are done.

For part (c) we simply note that $\log 1/|\zeta(s)| = -\Re \log \zeta(s)$, and so $1/\zeta(s) \ll \log|t|$ follows immediately from part (b).

2. Show that if $|t| \geq 4$ then the number of zeros of $\zeta(s)$ in the disc of radius r around $1+it$ is $O(r \log|t|)$ for all $r \leq 3/4$. *Hint: Again, use the formula for $\frac{\zeta'}{\zeta}$ in terms of its zeros and take real parts at $s = 1 + r + it$.*

Solution: Observe that if $r \leq c/\log|t|$ for some sufficiently small constant $c > 0$ then there are in fact no zeros in the disc of radius r around $1 + it$ by the zero-free region, so the estimate certainly holds for such small r. On the other hand, for large r, say $1/6 < r \leq 3/4$ we can use Jensen's inequality with the function $f(z) = \zeta(2 + it + z)$.

The tricky part is the region $c/\log|t| \leq r \leq 1/6$. As in the solution to 1a) the formula for $\frac{\zeta'}{\zeta}$ in terms of zeros yields

$$\sum_{\rho} \Re \frac{1}{s_1 - \rho} \ll \log|t|$$

where $s_1 = 1 + r + it$, where the sum is over all zeros ρ such that $|\rho - (3/2 + it)| \leq 5/6$. Each term is non-negative, and those zeros in the disc of radius r around $1 + it$ contribute at least $1/2r$, say, and so the number of zeros is $O(r \log|t|)$ as required.

3. If we arrange the non-trivial zeros of the Riemann zeta function in the upper half-plane as $\rho_n = \sigma_n + i\gamma_n$ where $0 < \gamma_1 \leq \gamma_2 \leq \cdots$ then show that

$$\gamma_n \sim \frac{2\pi n}{\log n}.$$

Deduce that $\sum_\rho \frac{1}{|\rho|} = \infty$.

Solution: We use the asymptotic formula

$$N(T) \sim \frac{1}{2\pi} T \log T.$$

In particular,

$$n = N(\gamma_n) = (1 + o(1)) \frac{1}{2\pi} \gamma_n \log \gamma_n.$$

In particular, for sufficiently large n, $\gamma_n \leq n$, and so $\log\log\gamma_n \ll \log\log n$. Taking logarithms

$$\log \gamma_n + O(1) + \log\log\gamma_n = \log n$$

and so $\log \gamma_n = (1 + o(1)) \log n$. It follows that

$$\gamma_n \sim \frac{2\pi n}{\log n}.$$

In particular, since $|\rho_n| \ll 1 + |\gamma_n| \ll n/\log n$,

$$\sum_{\rho} \frac{1}{|\rho|} \gg \sum_{n} \frac{\log n}{n} = \infty.$$

4. Show that there exists some constant $C > 0$ such that there is no vertical gap greater than C between successive zeros of $\zeta(s)$, that is,

$$N(T + C) - N(T) > 0$$

for all sufficiently large T.

Solution: Recall the asymptotic formula

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + R(T),$$

say, where $R(T) \ll \log T$. The first term is easier to deal with if we write

$$\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} = \frac{1}{2\pi} \int_0^T \log \frac{t}{2\pi} \, \mathrm{d}t.$$

If we take the difference then

$$N(T + C) - N(T) = \frac{1}{2\pi} \int_T^{T+C} \log \frac{t}{2\pi} \, \mathrm{d}t + R(T + C) - R(T).$$

The first term is trivially at least $\frac{C}{2\pi} \log \frac{T}{2\pi}$. Provided we choose C sufficiently large, and $T \geq C$, say, the error terms are in absolute value

$$|R(T + C) - R(T)| < \frac{C}{2\pi} \log \frac{T}{2\pi},$$

and the estimate follows.

5. Let $M(x) = \sum_{n \leq x} \mu(n)$.[1] Let $\Theta = \sup\{\sigma : \zeta(\sigma + it) = 0\}$.

(a) Show that $M(x) = \Omega_\pm(x^{\sigma_0})$ for every $\sigma_0 < \Theta$.

(b) If there is a simple zero of $\zeta(s)$ at $\rho = \Theta + it$ then show that $M(x) = \Omega_\pm(x^\Theta)$.

(c) If there is a zero of $\zeta(s)$ of multiplicity $m \geq 2$ at $\rho = \Theta + it$ then show that

$$M(x) = \Omega_\pm(x^\Theta (\log x)^{m-1}).$$

In particular, if we could prove that $M(x) = O(x^{1/2})$ then we'd get both the Riemann hypothesis and also that all zeros of $\zeta(s)$ are simple! *Hint: Consider the function* $\frac{1}{s\zeta(s)} - c\frac{(m-1)!}{(s-\Theta)^m}$ *for some constant* $c > 0$.

Solution: We mimic the proofs for $\psi(x)$ given in lectures.

For part (a), fix $\sigma_0 < \Theta$, and suppose that for all sufficiently large x, we have the estimate $M(x) \leq cx^{\sigma_0}$, where c is a constant. Consider the function

$$F(s) = \int_1^\infty (cx^{\sigma_0} - M(x))\, x^{-s-1} \, \mathrm{d}x.$$

This integral obviously converges absolutely in the half-plane $\sigma > 1$, and it is of the form $\int_1^\infty A(x) x^{-s} \, \mathrm{d}x$ where $A(x) \geq 0$ for all large enough x, so we can apply Landau's lemma.

In the half-plane $\sigma > 1$ we calculate that

$$F(s) = \frac{c}{s - \sigma_0} + \frac{1}{s\zeta(s)}.$$

This has a pole at $s = \sigma_0$, but otherwise there are no poles for real $s > 0$. By Landau's lemma there are no poles for any s with real part $> \sigma_0$, which contradicts the choice of σ_0, since by the definition of Θ there is a zero of $\zeta(s)$ wih real part $> \sigma_0$.

[1] Mertens conjectured in 1897 that $|M(x)| \leq x^{1/2}$ for all $x \geq 1$. This was disproved by Odlyzko and te Riele in 1984.

It follows that for any fixed $c > 0$ we have $M(x) > cx^{\sigma_0}$ for arbitrarily large x. Similarly $M(x) < -cx^{\sigma_0}$, and so $M(x) = \Omega_\pm(x^{\sigma_0})$ as required.

For part (b), suppose that there is a simple zero of $\zeta(s)$ at $\rho = \Theta + it$, and let $c > 0$ be a constant to be chosen later. Suppose that $M(x) \leq cx^\Theta$ for all large enough x, and consider the function

$$\int_1^\infty \left(cx^\Theta - M(x)\right)(1 + \cos(\theta - t\log x))x^{-s-1}\,\mathrm{d}x$$

where $\theta \in [0, 2\pi)$ will be chosen later. On one hand this is (in the half-plane $\sigma > 1$)

$$F(s) + \frac{e^{i\theta}F(s+it) + e^{-i\theta}F(s-it)}{2}.$$

Note that F has poles at Θ and at zeros of $\zeta(s)$. It follows that, on the real line, there are no poles for $s > \Theta$, and at $s = \Theta$ there is a pole with residue

$$c + \frac{1}{2c_1}\left(\frac{e^{i\theta}}{\Theta + it} + \frac{e^{-i\theta}}{\Theta - it}\right) = c + \Re\frac{e^{i\theta}}{c_1\rho},$$

where $\zeta(s) = c_1(s-\rho) + O((s-\rho)^2)$ for s close to ρ, and $c_1 \neq 0$ (since by assumption ρ is a simple zero).

On the other hand, as in the proof given in lectures, from the integral expression we see that the integral is $\geq -C$ for some absolute constant C as $s \to \Theta$ from the right along the real axis. It follows that the residue of the pole at Θ is > 0, and thus we have a contradiction if we choose θ such that $e^{i\theta}/c_1\rho = -1/|c_1\rho|$ and choose $c = 1/2\,|c_1\rho|$. Thus $M(x) \geq cx^\Theta$ for arbitrarily large x, and similarly $M(x) \leq -cx^\Theta$ for arbitrarily large x, and so $M(x) = \Omega_\pm(x^\Theta)$.

For part (c), suppose that $M(x) \leq cx^\Theta(\log x)^{m-1}$ for all sufficiently large x, where c will be chosen later. As the hint suggests, we consider the integral

$$F(s) = \int_1^\infty \left(cx^\Theta(\log x)^{m-1} - M(x)\right)x^{-s-1}\,\mathrm{d}x$$

which converges absolutely for $\sigma > 1$ to

$$\frac{c(m-1)!}{(s-\Theta)^m} - \frac{1}{s\zeta(s)}.$$

As above, if we multiply the integrand by $(1+\cos(\theta - t\log x))$ then we arrive at a function with non-negative integrand which has, for a suitable choice of θ and c, a pole of order m with negative residue at $s = \Theta$, which contradicts the fact that the integral stays bounded away from $-\infty$ as $s \to \Theta$ along the real axis. This contradiction, and a similar argument with the signs reversed, shows that $M(x) = \Omega_\pm(x^\Theta(\log x)^{m-1})$ as required.