Analytic Number Theory Sheet 2 - Solutions

Lent Term 2020

1. For a Dirichlet series $F(s) = \sum \frac{a_n}{n^s}$ let σ_c have the property that $F(s)$ converges for all s with $\sigma > \sigma_c$ and for no s with $\sigma < \sigma_c$. Let σ_a have the property that $F(s)$ converges *absolutely* if $\sigma > \sigma_a$ and does not converge absolutely if $\sigma < \sigma_a$. Show that

$$\sigma_c \leq \sigma_a \leq \sigma_c + 1.$$

Give examples to show that both $\sigma_c = \sigma_a$ and $\sigma_c + 1 = \sigma_a$ are possible.

Solution: It is trivial that $\sigma_c \leq \sigma_a$, since absolute convergence implies convergence. To show that $\sigma_a \leq \sigma_c + 1$ it suffices to show that if $\sigma > \sigma_c + 1$ then the Dirichlet series converges absolutely.

Let $\sigma = \sigma_c + 1 + 2\epsilon$. By assumption the series $\sum a_n n^{-\sigma_c - \epsilon}$ is convergent. In particular, the summands tend to zero, and are there bounded, and so $a_n \ll n^{\sigma_c + \epsilon}$. Therefore

$$\sum \frac{|a_n|}{|n^s|} = \sum \frac{|a_n| / n^{\sigma_c + \epsilon}}{n^{1+\epsilon}} \ll \sum \frac{1}{n^{1+\epsilon}} \ll 1,$$

and so the Dirichlet series converges absolutely at s as required, and $\sigma_a \leq \sigma_c + 1$.

The Riemann zeta function itself ($a_n \equiv 1$) shows that $\sigma_c = \sigma_a$ is possible. An example where $\sigma_c + 1 = \sigma_a$ is the similar function

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}.$$

Obviously $\sigma_a = 1$, so it suffices to show that this Dirichlet series converges whenever $\sigma > 0$. This can be justified by an appeal to one of several general analytic results (for example, the alternating series test, since it suffices to show that it converges for real $\sigma > 0$), but here's a direct demonstration using partial summation.

Partial summation gives that, with $A(x) = 1$ if $\lfloor x \rfloor$ is odd and 0 otherwise,

$$\sum_{1 \leq n \leq x} \frac{(-1)^{n-1}}{n^s} = \frac{A(x)}{x^s} + s \int_1^x \frac{A(x)}{x^{s+1}} \, \mathrm{d}x.$$

Taking the limit as $x \to \infty$ implies

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = s \int_1^{\infty} \frac{A(x)}{x^{s+1}} \, \mathrm{d}x.$$

Since $A(x) = O(1)$ this integral converges absolutely when $\sigma > 0$, and hence the sum converges and we're done. The alternating series test implies that this converges whenever $s > 0$ is real.

2. For fixed $\sigma \in \mathbb{R}$ let $\nu(\sigma)$ denote the infimum of those exponents ν such that $\zeta(\sigma + it) \ll |t|^{\nu}$ for all $|t| \geq 4$. (The Lindelöf hypothesis is the conjecture that $\nu(1/2) = 0$.)

(a) Show that $\nu(\sigma) = 0$ for $\sigma \geq 1$.

(b) Show that $\nu(\sigma) \leq 1 - \sigma$ for $0 < \sigma \leq 1$.

(c) Show that $\nu(\sigma) = \nu(1 - \sigma) + 1/2 - \sigma$, and in particular $\nu(\sigma) = 1/2 - \sigma$ for $\sigma \leq 0$. (You may use Stirling's approximation, that $|\Gamma(s)| \asymp t^{\sigma - 1/2} e^{-\pi t/2}$ as $t \to \infty$ for σ uniformly bounded.)

Solution: The Dirichlet series represention and the triangle inequality immediately imply that $|\zeta(\sigma+it)| \leq |\zeta(\sigma)| < \infty$. This demonstrates $\nu(\sigma) \leq 0$ for $\sigma > 1$. That $\nu(\sigma) \leq 0$ for $\sigma = 1$ is covered by part (b). We also need to show that $\nu(\sigma) \geq 0$ for $\sigma \geq 1$. For this, recall that for $\sigma > 1$ we have

$$\left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right) \zeta(s) = 1.$$

The left-hand side is at most $\zeta(\sigma)|\zeta(s)|$, and therefore $|\zeta(s)| \geq 1/\zeta(\sigma) \gg 1$. This shows $\nu(\sigma) \geq 0$ for $\sigma > 1$.

Showing that $\nu(1) \geq 0$ is a little tricky. It follows from the fact that $\nu(\sigma)$ is a continuous function, which is a simple consequence of the Phragmén-Lindelöf principle, a generalisation of the maximum modulus principle. Here is a direct proof using what we have shown in the course.

First note that by looking at the Dirichlet series when $\sigma > 1$

$$|\log \zeta(s)| \leq \sum \frac{\Lambda(n)}{\log n} n^{-\sigma} = \log \zeta(\sigma).$$

Therefore, since $|\zeta(\sigma)| \ll (\sigma-1)^{-1}$, if we choose $\sigma \approx 1 + 1/(\log T)^2$, say, then $|\log \zeta(\sigma+iT)| \leq 2\log\log T + O(1)$ (for some $T \geq 2$). Furthermore,

$$\log \zeta(\sigma+iT) - \log \zeta(1+iT) = \int_1^{\sigma} \frac{\zeta'}{\zeta}(\sigma+it)\,\mathrm{d}\sigma,$$

which is permissible since there are no zeros of $\zeta(s)$ on the line $\sigma = 1$. By Lemma 25 of the course there are arbitrarily large T such that the integrand is this is $\ll (\log T)^2$, and hence the integral overall is $O(1)$, and so $|\log \zeta(1+iT)| \leq 2\log\log T + O(1)$. Since $\log(1/|\zeta|) = -\Re \log \zeta$ it follows that $\log(1/|\zeta(1+iT)|) \leq 2\log\log T + O(1)$, and hence $|\zeta(1+iT)| \gg (\log T)^{-2}$. Since this holds for arbitrarily large T, it follows that $\nu(1) \geq 0$ as required.

For part (b) recall that Lemma 20 provided the estimate

$$\zeta(s) \ll (1 + |t|^{1-\sigma}) \log |t|$$

which holds uniformly for $0 < \delta \leq \sigma \leq 2$ and $|t| \geq 4$, say. This immediately implies that $\nu(\sigma) \leq 1-\sigma$ when $0 < \sigma \leq 1$.

Finally, we will use the functional equation for part (c). The functional equation implies that there is some constant $C_\sigma > 0$ such that

$$|\zeta(\sigma+it)| = C_\sigma |\sin(\pi s/2)| |\Gamma(1-\sigma-it)| |\zeta(1-\sigma-it)|.$$

Stirling's approximation as given in the hint implies that

$$|\zeta(\sigma+it)| \asymp_\sigma |\sin(\pi s/2)| t^{\sigma-1/2} e^{-\pi t/2} |\zeta(1-\sigma-it)|.$$

For any $\epsilon > 0$ the bound $|\zeta(1-\sigma-it)| \ll |t|^{\nu(1-\sigma)+\epsilon}$, and hence

$$|\zeta(\sigma+it)| \ll_{\sigma,\epsilon} |\sin(\pi s/2)| t^{\sigma-1/2+\nu(1-\sigma)+\epsilon} e^{-\pi t/2} \ll t^{\sigma-1/2+\nu(1-\sigma)+\epsilon}$$

and hence by definition $\nu(\sigma) \leq \nu(1-\sigma) + \sigma - 1/2$. Replacing σ by $1-\sigma$ also implies that $\nu(1-\sigma) \leq \nu(\sigma) + 1/2 - \sigma$, and hence $\nu(\sigma) = \nu(1-\sigma) + \sigma - 1/2$ as required.

3.

$$\sum_{\substack{a,b \geq 1 \\ (a,b)=1}} \frac{1}{a^2 b^2} = \frac{5}{2}.$$

Hint: Use the fact that $\sum_{d|n} \mu(d) = 0$ *if* $n > 1$.

Solution: Using the given hint, we can write

$$\sum_{\substack{a,b\geq 1\\(a,b)=1}} \frac{1}{a^2b^2} = \sum_{a,b\geq 1}\sum_{d\geq 1} 1_{d|a}1_{d|b}\frac{\mu(d)}{a^2b^2}.$$

Changing the order of summation and writing $a = dn$ and $b = dm$ this is

$$\sum_{d\geq 1}\frac{\mu(d)}{d^4}\sum_{n,m\geq 1}\frac{1}{n^2m^2} = \frac{\zeta(2)^2}{\zeta(4)}.$$

Using the values $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$ proved in lectures gives the result.

4. Sketch a proof that if $s \neq 1$ and $\zeta(s) \neq 0$ then, if x is not an integer,

$$\sum_{n\leq x}\frac{\Lambda(n)}{n^s} = \frac{x^{1-s}}{1-s} - \lim_{T\to\infty}\sum_{\substack{\rho\\|\gamma|\leq T}}\frac{x^{\rho-s}}{\rho-s} - \frac{\zeta'}{\zeta}(s) + \sum_{k=1}^{\infty}\frac{x^{-2k-s}}{2k+s}.$$

Solution: We can follow the proof of the explicit formula given in lectures very closely, just making the adjustment that we are considering the Dirichlet series with coefficients $\Lambda(n)/n^s$ instead of $\Lambda(n)$. This can be written as $-\frac{\zeta'}{\zeta}(u+s)$, and hence by Perron's formula,

$$\sum_{n\leq x}\frac{\Lambda(n)}{n^s} = -\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\frac{\zeta'}{\zeta}(u+s)\frac{x^u}{u}\,\mathrm{d}u$$

for some suitable σ_0. It is important to make a suitable choice of σ_0. We need it to be > 0 to apply Perron's formula, and also $> 1-\sigma$ so that the Dirichlet series converges absolutely. Because we aren't too concerned with the error terms, we have a lot of freedom over how we choose σ_0 (e.g. we don't need to worry about the $1/\log x$ factor in the proof from lectures, which was a result of trying to keep the error terms small). Since we're going to take the limit $T \to \infty$ the dependence on x can afford to be quite weak. A suitable choice is, for example, $\sigma_0 = \max(1, 2-\sigma)$.

The poles coming from $\frac{\zeta'}{\zeta}$ occur at $1-s$, $\rho - s$ for zeros of $\zeta(s)$ in the critical strip, and $-2k-s$ for $k \geq 1$ coming from the trivial zeros. By assumption none of these are zero, so they are not also poles of the x^u/u term, which allows for the residues to be calculated in the same fashion. The other pole is at $u = 0$ which has $\frac{\zeta'}{\zeta}(s)$ as a residue (again, note that is well-defined by assumption on s).

5.

(a) Using elementary methods show that

$$\sum_{n\leq x}\Lambda(n)\left\lfloor\frac{x}{n}\right\rfloor = x\log x + O(x).$$

(b) Deduce that

$$\sum_{n\leq x}\frac{\Lambda(n)}{n} \sim \log x.$$

Compare this to the result of Question 4 as $s \to 1$.

Solution: Using $\lfloor x \rfloor = \sum_{n \le x} 1$

$$\sum_{n \le x} \Lambda(n) \left\lfloor \frac{n}{x} \right\rfloor = \sum_{n \le x} \Lambda(n) \sum_{m \le x/n} 1 = \sum_{nm \le x} \Lambda(n) = \sum_{k \le x} 1 \star \Lambda(k) = \sum_{k \le x} \log k.$$

The right-hand side is $x \log x + O(x)$ by partial summation, which concludes part (a).

For part (b), replacing $\lfloor x/n \rfloor$ by x/n introduces an error of $O(\psi(x)) = O(x)$ by Chebyshev's estimates. It follows that

$$\sum_{n \le x} \frac{\Lambda(n)}{n} = \log x + O(1).$$

Consider the expression in question 4 for real $s > 1$, and what happens to each term as $s \to 1$ from the right. The term $x^{1-s}/1 - s$ approaches a pole at $s = 1$, around which it can be expanded as $1/(1-s) + \log x + O(s-1)$. This is cancelled out by the pole of $\zeta'/\zeta(s)$ which is $1/(1-s) + O(1)$ around $s = 1$. The final term is $\frac{1}{2} \log \left(\frac{1+1/x}{1-1/x} \right) - \frac{1}{x}$ for $x > 1$. This therefore implies that

$$O(1) = \sum_{\rho} \frac{x^{\rho - 1}}{\rho - 1} + \frac{1}{2} \log \left(\frac{1 + 1/x}{1 - 1/x} \right).$$

Since the zeros have a symmetry about $\rho \mapsto 1 - \rho$ in the critical strip by the functional equation we can write this as

$$O(1) = -\sum_{\rho} \frac{x^{-\rho}}{\rho} + \frac{1}{2} \log \left(1 + \frac{2}{x - 1} \right),$$

and hence for any $0 < y < 1$,

$$\sum_{\rho} \frac{y^{\rho}}{\rho} = O(1) + \frac{1}{2} \log \left(1 + \frac{2y}{1 - y} \right).$$

If we are a little more precise, and use the fact that $\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1)$ we in fact see that, for any $x > 1$,

$$\sum_{n \le x} \frac{\Lambda(n)}{n} = \log x + \sum_{\rho} \frac{x^{-\rho}}{\rho} - \gamma - \frac{1}{x} + \frac{1}{2} \log \left(\frac{x+1}{x-1} \right).$$

6.

(a) Show that if $\sigma > 0$ and $k \ge 1$ then

$$\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{y^s}{s^{k+1}} \, \mathrm{d}s = \begin{cases} \frac{(\log y)^k}{k!} & \text{if } y \ge 1 \text{ and} \\ 0 & \text{if } y \le 1. \end{cases}$$

(b) Give an explicit formula for any $k \ge 1$ for

$$\sum_{n \le x} \Lambda(n) \left(\log \frac{x}{n} \right)^k,$$

sketching a proof of the formula you give.

Solution: Part (a) is a simple adaptation of evaluation of the integral $\int_{\sigma - i\infty}^{\sigma + i\infty} \frac{y^s}{s} \, \mathrm{d}s$ as presented in lectures. The only differences are that the residue at the pole at $s = 0$ is now $(\log y)^k / k!$, and that the integral can now be evaluated at $y = 1$ for $k \ge 1$ because the integral over the 'error sides' of the rectangle decays sufficiently quickly. Alternatively, the case $y = 1$ can be deduced from the cases $y > 1$ and $y < 1$ since the right-hand side approaches the same limit 0 as y tends to 1 from the right or the left.

We can then use this to derive an explicit formula for the sum given. As usual, adapting Perron's formula in the obvious fashion,

$$\frac{1}{k!}\sum_{n\leq x}\Lambda(n)\left(\log\frac{x}{n}\right)^k = -\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\frac{\zeta'}{\zeta}(s)\frac{x^s}{s^{k+1}}\,\mathrm{d}s$$

for any $\sigma_0 > 1$. The pole at $s = 0$ contributes $\frac{-F^{(k)}(0)}{k!}$ where $F(s) = \frac{\zeta'}{\zeta}(s)x^s$.

The pole at $s = 1$ contributes x. The pole from each trivial zero $s = -2m$ contributes

$$-\frac{x^{-2m}}{(-2m)^{k+1}}.$$

Similarly, the pole from each zero in the critical strip contributes

$$-\frac{x^\rho}{\rho^{k+1}}.$$

Altogether, then,

$$\sum_{n\leq x}\Lambda(n)\left(\log(x/n)\right)^k = k!\left(x - \sum_\rho\frac{x^\rho}{\rho^{k+1}} - \frac{F^{(k)}(0)}{k!} - \sum_{m=1}^\infty\frac{x^{-2m}}{(-2m)^{k+1}}\right).$$

The term $F^{(k)}(0)$ can be expanded out if desired as

$$c_0(\log x)^k + c_1(\log x)^{k-1} + \cdots + c_k$$

where $c_0 = \frac{\zeta'}{\zeta}(0) = \log 2\pi$, for example, and $c_1 = k(\frac{\zeta'}{\zeta})'(0)$, and so on.