PRIMER ON COMPLEX ANALYSIS FOR ANALYTIC NUMBER THEORY

As is traditional in this area, we will often write $s = \sigma + it \in \mathbb{C}$ for an arbitrary complex variable, in which case $\sigma = \Re s$ is the real part of s and $t = \Im s$ is the imaginary part. The derivative of a function $f : \mathbb{C} \to \mathbb{C}$ at a point s is defined to be

$$f'(s) = \lim_{z \to s} \frac{f(z) - f(s)}{z - s}.$$

Implicit in this definition is the fact the limit exists and remains the same for any sequence of (z) which has s as a limit. A neighbourhood of s is a bounded open set which contains s. We say that f is holomorphic on an open set U if $f'(s)$ exists for every $s \in U$, and that f is holomorphic at s if f is holomorphic on some neighbourhood of s. A function is entire if it is holomorphic on $\mathbb{C}$.

A smooth curve is a continuous function $\gamma : [a, b] \to \mathbb{C}$ with a non-vanishing continuous derivative which is injective (except possibly at the endpoints). More generally, a contour is a finite sequence of smooth curves joined at the endpoints. The contour integral of f along a smooth curve γ is

$$\int_\gamma f(s)\,\mathrm{d}s = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t,$$

which is extended in the obvious fashion for general contours.

Theorem 1 (Cauchy's theorem)**.** *If U is an open simply connected set, f is holomorphic on U, and γ is a closed contour in U, then*

$$\int_\gamma f(s)\,\mathrm{d}s = 0.$$

Theorem 2 (Cauchy integral formula)**.** *If D is a closed disc with boundary circle C and f is holomorphic on a neighbourhood of D then for every a in the interior of D*

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s - a}\,\mathrm{d}s.$$

Theorem 3. *Every holomorphic function is analytic. That is, if f is holomorphic on some neighbourhood of a then there is some open disc centred at a in which f can be expanded as a convergent power series*

$$f(s) = \sum_{n=0}^{\infty} c_n (s - a)^n.$$

The coefficients c_n are

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - a)^{n+1}}\,\mathrm{d}w,$$

where C is any circle centred at a on and within which f is holomorphic.

Theorem 4 (Identity Theorem)**.** *If f and g are both holomorphic on an open and connected set D and $f = g$ for all $s \in S \subset D$, where S is such that there is some $x \in D$ such that every neighbourhood of x in D contains some point in S, then $f \equiv g$ on D.*

A function f is meromorphic at a point a if there is some neighbourhood of a on which either f or $1/f$ is holomorphic. In this case there is some n such that $(s - a)^n f(s)$ is holomorphic and non-zero in a neighbourhood of a. If $n > 0$ then a is a pole of f of order n. If $n < 0$ then a is a zero of f of order $-n$. If f is meromorphic at a then there is some neighbourhood of a in which f can be expressed as a Laurent series,

$$\sum_{n=-k}^{\infty} c_n (z - a)^n,$$

for some finite integer k. The coefficient c_{-1} is the residue of f at a, and is denoted by $\mathrm{Res}(f, a)$.

Theorem 5 (Residue theorem)**.** *If U is a simply connected open set which contains a simple closed curve γ, f is holomorphic on γ, and is holomorphic inside γ except for a finite sequence $a_1, \ldots, a_k$, then*

$$\int_\gamma f(s)\,\mathrm{d}s = 2\pi i \sum_{i=1}^{k} \mathrm{Res}(f, a_k).$$

Theorem 6 (Maximum modulus principle)**.** *If U is a connected open set and f is holomorphic on U, and if there exists some $a \in U$ such that $|f(a)| \geq |f(s)|$ for all s in a neighbourhood of a, then f is constant on D.*

If U is a simply connected open set and f is holomorphic and non-zero on U then we define $\log f(z)$ on U as

$$\log f(z) = a + \int_b^z \frac{f'(s)}{f(s)}\,\mathrm{d}s,$$

where $b \in U$ and a is such that $\exp(a) = f(b)$. The integral can be taken over any path between b and z. This function is well-defined up to a constant (depending on the choice of a and b) which is always an integral multiple of $2\pi i$.