Infinitely many zeros on the critical line

In this note we go through the solution of the bonus Question 6 on Sheet 3, which outlines a proof of Hardy's theorem, that there are infinitely many zeros on the line $\sigma = 1/2$. This proof is due to Pólya.

Recall that $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$. For $t \in \mathbb{R}$ define $\Xi(t) = \xi(\frac{1}{2} + it)$.

1. Show that (when $\sigma > 1$)

$$\Gamma(s/2)\pi^{-s/2}\zeta(s) = \int_0^\infty F(x)x^{\frac{s}{2}-1}\,\mathrm{d}x$$

where

$$F(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}.$$

We plug in the definition of $F(x)$ into the integral on the right-hand side, and interchange the sum and integral, so

$$\int_0^\infty F(x)x^{\frac{s}{2}-1}\,\mathrm{d}x = \sum_{n=1}^{\infty}\int_0^\infty x^{\frac{s}{2}-1}e^{-n^2\pi x}\,\mathrm{d}x.$$

We should justifty the interchange of sum and integral here. We can do this with the dominated convergence theorem, since the partial sums of $\sum_{n=1}^\infty x^{\frac{s}{2}-1}e^{-n^2\pi x}$ are bounded above in absolute value by $f(x) = \sum_{n=1}^\infty x^{\frac{\sigma}{2}-1}e^{-n^2\pi x}$, so it's enough to show that f is integrable. This in turn follows from the monotone convergence theorem, which implies that

$$\int_0^\infty f(x)\,\mathrm{d}x = \sum_{n=1}^{\infty}\int_0^\infty x^{\frac{\sigma}{2}-1}e^{-n^2\pi x}\,\mathrm{d}x = \sum_{n=1}^{\infty} n^{-\sigma}\pi^{-\sigma/2}\int_0^\infty x^{\frac{\sigma}{2}-1}e^{-x}\,\mathrm{d}x.$$

The integral here is just $\Gamma(\sigma/2)$ (which exists and is $< \infty$ for $\sigma > 0$) and so we are done provided $\sum n^{-\sigma} < \infty$, which holds when $\sigma > 1$.
Doing the same change of variable as above, we have that

$$\int_0^\infty x^{\frac{s}{2}-1}e^{-n^2\pi x}\,\mathrm{d}x = n^{-s}\pi^{-s/2}\int_0^\infty x^{\frac{s}{2}-1}e^{-x}\,\mathrm{d}x = n^{-s}\pi^{-s/2}\Gamma(s/2),$$

which is true provided $\sigma > 0$. Therefore, the claim follows, summing over all n.

2. Using

$$2F(x) + 1 = x^{-1/2}\left(2F(1/x) + 1\right) \tag{1}$$

(which is an application of Poisson summation) show that

$$\Gamma(s/2)\pi^{-s/2}\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty F(x)\left(x^{\frac{s}{2}-1} + x^{-\frac{1}{2}-\frac{s}{2}}\right)\mathrm{d}x$$

and deduce the functional equation.

We first note that since the integral on the right-hand side is convergent for all s (when $\sigma > 1$ the $x^{\frac{s}{2}-1}$ term dominates and we can argue as above, when $\sigma < 1$ the $x^{-\frac{1}{2}-\frac{s}{[}2}$ term dominates and we argue the same way with s replaced by $1-s$, and when $\sigma = 1$ it comes down to the convergence of

$$\sum_{n=1}^{\infty} \int_1^{\infty} (x^{-\frac{1}{2}} + x^{-1}) e^{-n^2 \pi x} \, \mathrm{d}x \ll \sum_{n=1}^{\infty} n^{-2} \int_1^{\infty} e^{-x} \, \mathrm{d}x < \infty.$$

In particular, the right-hand side is a meromorphic function in all of $\mathbb{C}$, with the only poles at $s = 0$ and $s = 1$. Therefore if we can show that the required identity holds when $\sigma > 1$, then by uniqueness of analytic continuation it must hold for all s.
When $\sigma > 1$ we can use the identity from part (1), to write

$$\Gamma(s/2)\pi^{-s/2}\zeta(s) = \int_0^{\infty} F(x) x^{\frac{s}{2}-1} \, \mathrm{d}x.$$

Divide this integral into the contribution from $[0, 1]$ and from $[1, \infty)$. The part from $[0, 1]$ we can write, using (1), as

$$\int_0^1 F(x) x^{\frac{s}{2}-1} \, \mathrm{d}x = \int_0^1 x^{\frac{s}{2}-1} \left(x^{-1/2} F(1/x) + \frac{x^{-1/2}}{2} - \frac{1}{2} \right) \mathrm{d}x.$$

We have

$$\frac{1}{2} \int_0^1 x^{\frac{s}{2}-3/2} \, \mathrm{d}x = \frac{1}{s-1} \quad \text{and} \quad \frac{1}{2} \int_0^1 x^{\frac{s}{2}-1} \, \mathrm{d}x = \frac{1}{s}$$

and so

$$\int_0^1 F(x) x^{\frac{s}{2}-1} \, \mathrm{d}x = \frac{1}{s-1} - \frac{1}{s} + \int_0^1 x^{\frac{s}{2}-\frac{3}{2}} F(1/x) \, \mathrm{d}x.$$

Making the change of variable $x = 1/y$ and adding back in the contribution from $[1, \infty)$ yields

$$\Gamma(s/2)\pi^{-s/2}\zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} F(x) \left(x^{\frac{s}{2}-1} + x^{-\frac{1}{2}-\frac{s}{2}} \right) \mathrm{d}x$$

as required. Since the right-hand side is the same if we replace s by $1-s$ the left-hand side must be also, and so

$$\pi^{-s/2}\zeta(s)\Gamma(s/2) = \pi^{-1/2+s/2}\Gamma((1-s)/2)\zeta(1-s),$$

which implies the functional equation by the reflection formula

$$\Gamma(s/2) = \frac{\pi}{\sin(\pi s/2)\Gamma(1-s/2)}$$

and duplication formula

$$\Gamma(1-s/2) = 2^s \pi^{1/2} \Gamma(1-s)/\Gamma((1-s)/2),$$

which together yield

$$\pi^{-s/2}\zeta(s) \frac{\pi^{1/2}}{\sin(\pi s/2) 2^s \Gamma(1-s)} \Gamma((1-s)/2) = \pi^{-1/2+s/2}\Gamma((1-s)/2)\zeta(1-s),$$

and hence

$$\zeta(s) = \pi^{s-1} 2^s \sin(\pi s/2) \Gamma(1-s) \zeta(1-s)$$

as required.

3. Use (1) to also show that $4F'(1) + F(1) = -1/2$.

Differentiating both sides of (1) yields

$$2F'(x) = -\frac{1}{2}x^{-3/2}(2F(1/x) + 1) - 2x^{-5/2}F'(1/x).$$

Thus setting $x = 1$ we have

$$2F'(1) = -\frac{1}{2}(2F(1) + 1) - 2F'(1).$$

Rearranging this yields the result.

4. Use integration by parts and set $x = e^{2u}$ to deduce that

$$\Xi(t) = 2\int_0^\infty \Phi(u)\cos(ut)\,\mathrm{d}u$$

where

$$\Phi(u) = 6e^{\frac{5}{2}u}F'(e^{2u}) + 4e^{\frac{9}{2}u}F''(e^{2u}).$$

Multiplying both sides of the result of part (2) by $\frac{1}{2}s(s-1)$ we have

$$\xi(s) = \frac{1}{2} + \frac{1}{2}s(s-1)\int_1^\infty F(x)\left(x^{\frac{s}{2}-1} + x^{-\frac{1}{2}-\frac{s}{2}}\right)\,\mathrm{d}x.$$

In particular, since $\Xi(t) = \xi(\frac{1}{2} + it)$, we have

$$\Xi(t) = \frac{1}{2} - \frac{1}{2}(t^2 + 1/4)\int_1^\infty F(x)\left(x^{i\frac{t}{2}-\frac{3}{4}} + x^{-it-\frac{3}{4}}\right)\,\mathrm{d}x.$$

Simplifying slightly and replacing x by e^{2u}, we have

$$\Xi(t) = \frac{1}{2} - 2(t^2 + 1/4)\int_0^\infty e^{\frac{u}{2}}F(e^{2u})\cos(ut)\,\mathrm{d}u.$$

One application of integration by parts yields

$$\int_0^\infty e^{\frac{u}{2}}F(e^{2u})\cos(ut)\,\mathrm{d}u = -\int_0^\infty \frac{1}{t}\sin(ut)\left(\frac{1}{2}e^{\frac{u}{2}}F(e^{2u}) + 2e^{\frac{5u}{2}}F'(e^{2u})\right)\,\mathrm{d}u.$$

Applying integration by parts again, and using the identity in part (3), yields

$$-\frac{1}{4t^2} - \frac{1}{t^2}\int_0^\infty \cos(ut)\left(\frac{1}{4}e^{\frac{u}{2}}F(e^{2u}) + e^{\frac{5u}{2}}F'(e^{2u}) + 5e^{\frac{5u}{2}}F'(e^{2u}) + 4e^{\frac{9u}{2}}F''(e^{2u})\right)\,\mathrm{d}u.$$

Simplifying this, if I is the original integral, this integration by parts shows that

$$I = -\frac{1}{4t^2} - \frac{1}{t^2}\left(\frac{I}{4} + \int_0^\infty \cos(ut)\left(6e^{\frac{5u}{2}}F'(e^{2u}) + 4e^{\frac{9u}{2}}F''(e^{2u})\right)\,\mathrm{d}u\right),$$

which, after rearranging and recalling the definition of $\Phi(u)$, yields

$$(t^2 + 1/4)I = -\frac{1}{4} - \int_0^\infty \cos(ut)\Phi(u)\,\mathrm{d}u.$$

Inserting this into the expression for $\Xi(t)$ above yields the result.

5. Deduce that for any $n \geq 0$

$$\Phi^{(2n)}(u) = \frac{(-1)^n}{\pi} \int_0^\infty \Xi(t) t^{2n} \cos(ut)\, \mathrm{d}t.$$

Taking the Fourier transform of both sides, we see that

$$\Xi(t) = 2 \int_0^\infty \Phi(u) \cos(ut)\, \mathrm{d}u$$

implies

$$\Phi(u) = \frac{1}{\pi} \int_0^\infty \Xi(t) \cos(ut)\, \mathrm{d}t.$$

In particular it follows immediately that for any $n \geq 1$, we have

$$\Phi^{(2n)}(u) = \frac{(-1)^n}{\pi} \int_0^\infty \Xi(t) t^{2n} \cos(ut)\, \mathrm{d}t,$$

provided we can 'differentiate under the integral sign'. By Leibniz's rule this is permitted since the relevant derivatives exist and are continuous, and the integrals converge absolutely by the rapid decay of $\Xi(t)$.

6. Noting that, since $F(x)$ is analytic for $\Re x > 0$, we know that $\Phi(u)$ is analytic for $-\frac{\pi}{4} < \Im(u) < \frac{\pi}{4}$, deduce that for $|u| < \pi/4$

$$\Phi(iu) = \sum_{n \geq 0} c_n u^{2n}$$

where

$$c_n = \frac{1}{\pi (2n)!} \int_0^\infty \Xi(t) t^{2n}\, \mathrm{d}t.$$

First note that by part (5)

$$c_n = \frac{(-1)^n}{(2n)!} \Phi^{(2n)}(0).$$

Since $\Phi(u)$ is analytic around z for $|z| < \pi/4$, we can write $\Phi(z)$ as a Taylor series,

$$\Phi(z) = \sum_{m \geq 0} \Phi^{(m)}(0) \frac{z^m}{m!}.$$

Since $\Phi(z)$ is an even function the even derivatives are all odd, and in particular $\Phi^{(m)}(0) = 0$ for odd m. The conclusion then follows setting $z = iu$.

7. Deduce from (1) that $\frac{1}{2} + F(x)$ and all its derivatives tend to zero as $x \to i$ provided the argument of $x - i$ is at most $\pi/2$ in absolute value.

Let $x = i + z$, say. We have

$$F(i+z) = \sum_{n=1}^{\infty} e^{-n^2\pi(i+z)} = \sum_{n=1}^{\infty} (-1)^n e^{-n^2\pi z} = 2F(4z) - F(z).$$

Using (1) we have

$$F(i+z) = z^{-1/2}F(1/4z) - z^{-1/2}F(1/z) - 1/2.$$

Therefore,

$$\frac{1}{2} + F(i+z) = z^{-1/2}F(1/4z) - z^{-1/2}F(1/z).$$

Provided the argument of z is at most $\pi/2$ in absolute value the right-hand side tends to 0 as $z \to 0$, by the rapid decay of F.

8. Deduce that $\Phi(iu)$ and all its derivatives tend to 0 as $u \to \pi/4$ along the real axis.

By definition

$$\Phi(iu) = 6e^{\frac{5}{2}iu}F'(e^{2ui}) + 4e^{\frac{9}{2}iu}F''(e^{2ui}).$$

It therefore suffices to show that $\frac{1}{2} + F(e^{2ui})$ and all of its derivatives tend to 0 as $u \to \pi/4$ along the real axis. This is precisely the content of part (7).

9. Deduce that the coefficients c_n must be both ≥ 0 and ≤ 0 infinitely often.

By part (6)

$$\Phi(iu) = \sum_{n \geq 0} c_n u^{2n}.$$

If $c_n = 0$ for all large n then we are done. Otherwise, suppose that c_n is all the same sign (say > 0) for all $n \geq k$. Taking kth derivatives, it follows that there are other constants $c_n' > 0$ such that

$$\Phi^{(2k)}(iu) = \sum_{n \geq k} c_n' u^{2n-2k}.$$

Since the left-hand side tends to 0 as $u \to \pi/4$ along the real axis (from above, say), and $c_n' > 0$, this is a contradiction.

10. Show that if $\Xi(t) > 0$ for $t > T$ then

$$\int_0^\infty \Xi(t) t^{2n}\,\mathrm{d}t > (T+1)^{2n} \int_{T+1}^{T+2} \Xi(t)\,\mathrm{d}t - T^{2n} \int_0^T |\Xi(t)|\,\mathrm{d}t.$$

By considering the worst possible contribution from $[0, T]$, we have

$$\int_0^\infty \Xi(t) t^{2n}\,\mathrm{d}t \geq \int_T^\infty \Xi(t) t^{2n}\,\mathrm{d}t - T^{2n} \int_0^T |\Xi(t)|\,\mathrm{d}t.$$

The contribution from $[T+1, T+2]$ is at least $(T+1)^{2n} \int_{T+1}^{T+2} \Xi(t)\,\mathrm{d}t$, and the contribution from $[T, T+1]$ and $[T+2, \infty)$ is > 0 by the assumed positivity of $\Xi(t)$.

11. Deduce that $\Xi(t)$ has infinitely many real zeros, and hence $\zeta(s)$ has infinitely many zeros on the line $\sigma = 1/2$.

Suppose that $\Xi(t)$ has only finitely many real zeros. Therefore, for sufficiently large t, either $\Xi(t) > 0$ or $\Xi(t) < 0$ always. Suppose that $\Xi(t) > 0$ for $t > T$ (the case $\Xi(t) < 0$ is similar). Then, by part (10),

$$\pi(2n)! c_n = \int_0^\infty \Xi(t) t^{2n}\,\mathrm{d}t > (T+1)^{2n} \int_{T+1}^{T+2} \Xi(t)\,\mathrm{d}t - T^{2n} \int_0^T |\Xi(t)|\,\mathrm{d}t.$$

In particular, (since T is fixed) the right-hand side is > 0 for sufficiently large n, and hence $c_n > 0$ for all sufficiently large many n, contradicting part (9).
Thus $\Xi(t)$ has infinitely many real zeros, and since

$$\Xi(t) = \xi(\tfrac{1}{2} + it) = \frac{1}{2}(t^2 + 1/4)\pi^{-\frac{1}{4} - it/2}\Gamma(\tfrac{1}{4} + \tfrac{t}{2}i)\zeta(1/2 + it),$$

and the Gamma function has no zeros for $\sigma > 0$, whenever $\Xi(t) = 0$ (for real t) we must have $\zeta(1/2 + it) = 0$, and ζ has infinitely many zeros on the line $\sigma = 1/2$ as required.