Analytic Number Theory Sheet 4

Lent Term 2020

1. Show that for $\sigma > 1/2$
$$\int_0^T |\zeta(\sigma+it)|^4 \, \mathrm{d}t \sim \frac{\zeta(2\sigma)^4}{\zeta(4\sigma)} T.$$

2. (a) Show that if F is a smooth function on $[0,1]$ then
$$|F(1/2)| \leq \int_0^1 (|F(t)| + \tfrac{1}{2}|F'(t)|) \, \mathrm{d}t.$$

   (b) Let $t_1, \ldots, t_R \in [1/2, T-1/2]$ be a set of points such that whenever $i \neq j$ we have $|t_i - t_j| \geq 1$. Show that for any smooth $F : [0,T] \to \mathbb{C}$ we have
$$\sum_{1 \leq i \leq R} |F(t_r)|^2 \leq \int_0^T \left(|F(t)|^2 + |F(t)F'(t)| \right) \mathrm{d}t.$$

   (c) Deduce that for any $a_n \in \mathbb{C}$ we have
$$\sum_{1 \leq i \leq R} \left| \sum_{n \leq N} a_n n^{it_r} \right|^2 \ll (T+N) \log N \sum_{n \leq N} |a_n|^2 .$$

3. By adapting the proof of Ingham given in lectures, show that if $c > 0$ is a constant such that $\zeta(\frac{1}{2}+iT) \ll T^c$ for all $T \geq 2$ then
$$N(\sigma, T) \ll T^{(2+4c)(1-\sigma)} (\log T)^{O(1)}$$
uniformly for $1/2 \leq \sigma \leq 1$. In particular, the Lindelöf hypothesis (that $\zeta(\frac{1}{2}+iT) \ll_\epsilon T^\epsilon$ for all $\epsilon > 0$) implies the Density Conjecture.

4. In this question we sketch an alternative approach to obtaining zero density estimates. Let $M(s) = \sum_{n \leq X} \frac{\mu(n)}{n^s}$, and for $1/2 \leq \alpha \leq 1$ let
$$R(\alpha) = \{\sigma + it : \alpha \leq \sigma \leq 1 \text{ and } T < t \leq 2T\}.$$

   (a) Show that if $a_n = \sum_{d|n} \mu(d) 1_{n/d \leq T} 1_{d \leq X}$ then for all $s \in R(\alpha)$
$$\zeta(s)M(s) = \sum_{n \leq TX} \frac{a_n}{n^s} + O(T^{-\alpha} X^{1-\alpha} \log X).$$

   (b) Show that if we choose $X^{1-\alpha} \leq T^\alpha (\log T)^{-2}$ then, if $s \in R(\alpha)$ is a zero of $\zeta(s)$, for some $X \leq N \leq TX$ we have
$$\left| \sum_{N < n \leq 2N} \frac{a_n}{n^s} \right| \gg \frac{1}{\log T}.$$

(c) After making a suitable choice of X, combine the result of part (b) with the mean-value estimate of question 1 to deduce the zero density estimate

$$N(\alpha, T) \ll T^{4\alpha(1-\alpha)}(\log T)^{O(1)}.$$

5. Fix some $\sigma > 1$.

(a) Show that for all t

$$|\zeta(\sigma + it)| \leq \zeta(\sigma).$$

(b) Show that for any $N \geq 1$ and $t \geq 0$

$$|\zeta(\sigma + it)| \geq \sum_{n=1}^{N} \frac{\cos(t \log n)}{n^\sigma} - \sum_{n>N} \frac{1}{n^\sigma}.$$

(c) Show that, for any $a_1, \ldots, a_N \in \mathbb{R}$ and $\epsilon > 0$ there exist arbitrarily large t such that there exist $m_1, \ldots, m_N \in \mathbb{N}$ with

$$|ta_n - m_n| \leq \epsilon$$

for $1 \leq n \leq N$.

(d) Show that, for any $\epsilon > 0$, there are arbitrarily large t such that

$$|\zeta(\sigma + it)| \geq (1 - \epsilon)\zeta(\sigma).$$