# Analytic Number Theory Sheet 4

Lent Term 2019

Questions 3 and 4 will be marked.

1. Show that if $f : \mathbb{N} \to \mathbb{C}$ is a multiplicative function with period $q$ and $f(n) = 0$ whenever $(n, q) > 1$ then $f$ is a Dirichlet character modulo $q$.

2. (a) Show that, if $\chi$ is a non-principal Dirichlet character modulo $q$, then

$$\sum_{n\leq x} \frac{\chi(n)\log n}{n} = L(1,\chi)\sum_{d\leq x}\frac{\Lambda(d)\chi(d)}{d} + O_\chi\left(\frac{1}{x}\sum_{d\leq x}\Lambda(d)\right).$$

(b) Show that also

$$\sum_{n\leq x}\frac{\chi(n)\log n}{n} = -L'(1,\chi) + O_q\left(\frac{\log x}{x}\right).$$

(c) Deduce that

$$\sum_{n\leq x}\frac{\chi(n)\Lambda(n)}{n} \ll_\chi 1.$$

(d) Deduce further that, if $(a, q) = 1$, then

$$\sum_{\substack{p\leq x\\ p\equiv a \pmod q}} \frac{\log p}{p} = \frac{\log x}{\phi(q)} + O_q(1).$$

3. (a) Show that if $\chi$ is a non-principal character modulo $q$ then

$$\sum_{n>x}\frac{\chi(n)}{n^{1/2}} \ll_\chi \frac{1}{x^{1/2}}.$$

(b) Show that, if $r(n) = \sum_{d|n}\chi(d)$, then

$$\sum_{n\leq x}\frac{r(n)}{n^{1/2}} = 2x^{1/2}L(1,\chi) + O_\chi(1).$$

(c) Show that if $\chi$ is a quadratic character then $L(1, \chi) > 0$.

4. (a) Show that, if $\chi_1$ and $\chi_2$ are two Dirichlet characters of moduli $q_1$ and $q_2$ respectively, and if $\chi_1\chi_2$ is non principal, then $L(s, \chi_1)L(s, \chi_2)$ has at most one real zero $\beta$ such that

$$1 - \frac{c}{\log(q_1q_2)} < \beta < 1,$$

where $c > 0$ is some absolute constant.

(b) Deduce that there is some $c > 0$ such that if $(q_i)$ is a strictly increasing sequence such that for each $q_i$ there is an exceptional zero $\beta_i > 1 - c/(\log q_i)$ then $q_{i+1} > q_i^2$.

(c) Deduce that, if we define an exceptional zero $\beta$ to be one such that $\beta > 1 - c/\log q$, for some sufficiently small $c > 0$, then the number of $q$ with an exceptional zero in $[1, Q]$ is $O(\log \log Q)$.

5. Let $a$ be a fixed non-zero integer. Show that the number of primes $p \leq x$ such that $p + x$ is square-free is $\sim c(a)\text{li}(x)$ for some explicit constant $c(a) > 0$.