Analytic Number Theory Sheet 3

Lent Term 2019

Questions 3 and 5 will be marked.

1. Assuming the Riemann hypothesis prove that
$$\psi(x) = x + O(x^{1/2+o(1)}).$$

2. Prove that
$$\pi(x) = \mathrm{li}(x) + \Omega_-\left(\frac{x^{1/2}}{\log x}\right),$$
where $\mathrm{li}(x) = \int_2^x \frac{1}{\log t}\,\mathrm{d}t$. Why is establishing a Ω_+ result harder?

3. (a) Show that if $\sigma > 1 - c/\log t$ for some small constant c and $|t| \geq 7/8$ then
$$\frac{1}{\zeta(s)} \ll \log t.$$
   (b) Show that if $1 - c/\log t < \sigma \leq 2$ and $|t| \leq 7/8$ then $1/\zeta(s) \ll |s-1|$.
   (c) Show that
$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0.$$
   (Hint: use contour integration on the function $1/\zeta(s+1)$, taking a suitable contour around $s = 0$).

4. For fixed $\sigma \in \mathbb{R}$ let $\nu(\sigma)$ denote the infimum of those exponents ν such that $\zeta(\sigma + it) \ll |t|^{\nu}$ for all $|t| \geq 4$. (The unproven Lindelöf hypothesis is the statement that $\nu(1/2) = 0$.)
   (a) What is $\nu(\sigma)$ for $\sigma \geq 1$?
   (b) What is $\nu(\sigma)$ for $\sigma \leq 0$? (Use the functional equation.)

5. Let $M(x) = \sum_{n \leq x} \mu(n)$. Mertens conjectured in 1897 that $|M(x)| \leq x^{1/2}$ for all $x \geq 1$. This was disproved by Odlyzko and te Riele in 1984.
   (a) Show that $M(x) = \Omega_{\pm}(x^{1/2})$.
   (b) Show that, if $\zeta(s)$ has a multiple zero (i.e. a zero of order $m \geq 2$), then $M(x) = \Omega_{\pm}(x^{1/2}\log x)$, and hence Mertens conjecture is false (under this assumption).