Analytic Number Theory Sheet 2

Lent Term 2019

Questions 3 and 4 will be marked.

1. Show that, for any $1 \leq x, y$ and integer q,

$$\#\{x < n \leq x+y : (n,q)=1\} = \frac{\phi(q)}{q}y + O(2^{\omega(q)}).$$

2. Using sieve ideas show that

$$\#\{m^2+1 \leq x : m^2+1 \text{ is squarefree}\} \sim cx^{1/2}$$

where

$$c = \prod_{p \equiv 1 \pmod 4}\left(1 - \frac{2}{p^2}\right).$$

You may use without proof the fact that for fixed k the number of solutions to $m^2+1 = d^2k \leq x$ is $O(\log x)$.

3. For $(a,q)=1$ we denote by $\pi(x;q,a)$ the number of primes $\leq x$ which are congruent to a modulo q. Using Selberg's sieve prove that, for any $x \geq 0$ and $y \geq 2q$,

$$\pi(x+y;q,a) - \pi(x;q,a) \ll \frac{y}{\phi(q)\log(y/q)},$$

where the implied constant is absolute (i.e. independent of x, y, and q).

4. (a) Using Selberg's sieve prove that for any integer N the number of representations of N as the sum of two primes $p+q=N$ is $O(N^2/\phi(N)(\log N)^2)$.

   (b) Deduce that a positive proportion of all even integers can be written as the sum of two primes, in the sense that if A is the set of all such numbers then $\liminf_{N\to\infty} |A \cap [1,N]|/N > 0$. (Hint: Use the Cauchy-Schwarz inequality.)

5. Let $F_i(n) = a_i n + b_i$ be distinct linear forms with integer coefficients, each with positive leading coefficient. Suppose that there is no prime p such that $p \mid F_1(n)\cdots F_r(n)$ for all n. Use Brun's pure sieve to show that

$$\#\{n \leq x : F_1(n),\ldots,F_r(n) \text{ are all prime}\} \ll_{F_1,\ldots,F_r} \frac{x}{(\log x)^r}(\log\log x)^r.$$