Analytic Number Theory Sheet 1

Lent Term 2020

1. Let $\tau_3(n) = \sum_{a_1 a_2 a_3 = n} 1 = 1 \star \tau(n)$. Prove that

$$\sum_{n \leq x} \tau_3(n) = \frac{1}{2} x (\log x)^2 + c_1 x \log x + c_2 x + O(x^{2/3} \log x)$$

for some constants c_1 and c_2.

2. Let $\omega(n)$ count the number of distinct prime divisors of n.

   (a) Prove that

$$\sum_{n \leq x} \omega(n) = x \log \log x + O(x).$$

   (b) Prove the 'variance bound'

$$\sum_{n \leq x} |\omega(n) - \log \log x|^2 \ll x \log \log x.$$

   (c) Deduce that

$$\sum_{n \leq x} |\omega(n) - \log \log n|^2 \ll x \log \log x.$$

   and hence 'almost all n have $(1 + o(1)) \log \log n$ distinct prime divisors' in the sense that the number of $n \leq x$ such that $|\omega(n) - \log \log n| > (\log \log n)^{3/4}$ is $o(x)$.

3. (a) Show that

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma - \frac{\{x\} - 1/2}{x} + O(x^{-2}).$$

   (b) Let $\Delta(x)$ be the error term in the approximation for the sum of the divisor function, so that

$$\sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + \Delta(x).$$

   We proved in lectures that $\Delta(x) = O(x^{1/2})$. Prove the more precise estimate

$$\Delta(x) = x^{1/2} - 2 \sum_{a \leq x^{1/2}} \left\{ \frac{x}{a} \right\} + O(1).$$

   (c) Deduce that

$$\int_0^x \Delta(t) \, \mathrm{d}t \ll x$$

   (so that, 'on average', $\Delta(x) = O(1)$).

4. Prove the following Dirichlet series identities, and give for each a half-plane in which the identity is valid.

(a)

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \zeta(s)\zeta(s-1)$$

where $\sigma(n) = \sum_{d|n} d$,

(b)

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}$$

where $\lambda(n)$ is the completely multiplicative function such that $\lambda(p) = -1$ for all primes p,

(c)

$$\sum_{n=1}^{\infty} \frac{\tau(n)^2}{n^s} = \frac{\zeta(s)^4}{\zeta(2s)},$$

(d) and

$$\sum_{n=1}^{\infty} \frac{s(n)}{n^s} = \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)}$$

where $s(n)$ is the indicator function for the square-full numbers, i.e.

$$s(n) = \begin{cases} 1 & \text{if } p \mid n \text{ implies } p^2 \mid n \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

5. (a) Show that for $0 < \sigma < 1$

$$\zeta(s)\Gamma(s) = \int_0^{\infty} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) x^{s-1} \, \mathrm{d}x.$$

(b) Show that for $-1 < \sigma < 0$

$$\zeta(s)\Gamma(s) = \int_0^{\infty} \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} \right) x^{s-1} \, \mathrm{d}x.$$

(c) Deduce the functional equation for $\zeta(s)$, using the identity

$$\frac{1}{e^x - 1} = \frac{1}{x} - \frac{1}{2} + 2x \sum_{n=1}^{\infty} \frac{1}{4n^2\pi^2 + x^2}.$$