Additive Combinatorics Sheet 4

Lent Term 2021

Instructions

- These exercises are concerned with the material developed in Chapter 4: arithmetic progressions, Freiman isomorphisms, Bogolyubov-Ruzsa lemmas, and the Freiman-Ruzsa-Sanders inverse theorems and applications.
- There are 8 exercises, of varying (and non-monotone) difficulty and length. **You are not expected to do them all**, but I have provided 8 for the enthusiast. If you have solved any 4 then this should be sufficient evidence (for yourself) that are you where you should be.
- The examples class will be run by Aled Walker, who will mark before the class your solutions to 2 exercises.
- The two exercises to be marked should be submitted by **9am Monday 3rd May**. The class is **3:30pm Wednesday 5th May**. You should only submit the two solutions to be marked.
- Dr. Walker would appreciate knowing in advance of the class which exercises you found the most challenging. To this end, please submit the Self-Assessment form on Moodle before the class (even if you have not submitted any work to be marked).

1. Let
$$P = \{a + n_1 v_1 + \cdots + n_d v_d : 0 \leq n_i < N_i\}$$
be a generalised arithmetic progression of rank d in $\mathbb{Z}$. Let
$$\Gamma = \{(m_1, \ldots, m_d) \in \mathbb{Z}^d : m_1 v_1 + \cdots + m_d v_d = 0 \text{ and } |m_i| < N_i\}.$$
Recall that the volume of P is defined to be $\mathrm{vol}(P) = \prod_i N_i$. Show that
$$\frac{\mathrm{vol}(P)}{|P|} \leq |\Gamma| \leq 3^d \frac{\mathrm{vol}(P)}{|P|}.$$

2. Let P be a proper generalised arithmetic progression of rank d, and suppose $X \subset P$ has size $|X| \leq \epsilon |P|$. Show that $P \backslash X$ contains a proper generalised arithmetic progression Q of rank d with $|Q| \geq \epsilon^{-1} C^{-d}$ for some constant C.

3. (a) Show that for any $s \geq 1$ and $d \geq 1$ every finite subset of $\mathbb{Z}^d$ is Freiman s-isomorphic to a subset of $\mathbb{Z}$.
   (b) Hence deduce a Freiman-Ruzsa-Sanders inverse result for subsets of $\mathbb{Z}^d$ with small doubling.

4. (a) Suppose that $A \subset \{1, \ldots, N\}$ is such that $|A| = n \leq \log\log N$. Show that, if n is sufficiently large, then there exists a prime p
$$p \ll n^4 \log N \log\log N$$
and integer $t \neq 0$ such that all elements of $t \cdot A$ are distinct modulo p and are congruent to an integer in $(-p/4, p/4)$ modulo p and p divides no non-zero element of $A + A - A - A$.

[*Hint: Use the fact that every integer in* $(A+A-A-A)\backslash\{0\}$ *trivially has* $O(\log N)$ *many distinct prime divisors and consider* $(t\cdot a)_{a\in A}$ *modulo some suitable prime as* t *ranges over the interval* $[1,4^n+1]$. *You might also find useful that for all sufficiently large* X *there are at least* $\frac{X}{2\log X}$ *many primes in* $(X,2X]$.]

(b) Hence deduce that every finite set $A\subset\mathbb{Z}$ is 2-isomorphic to some $A'\subset\{1,\ldots,N\}$ where $N\leq C^{C^{|A|}}$ for some constant $C>1$.

5. Let G be a finite abelian group of odd order and let $A\subset G$ be a set of density $\alpha=|A|/|G|$. For any $0<\eta\leq 1$ let
$$\Delta_\eta(A)=\{\gamma\in\widehat{G}:|\widehat{1_A}(\gamma)|\geq\eta|A|\}.$$

   (a) Show that there are $c_\gamma\in\mathbb{C}$ with $|c_\gamma|=1$ such that for any $\Delta\subseteq\Delta_\eta(A)$ if
$$f(x)=\sum_{\gamma\in\Delta}c_\gamma\overline{\gamma(x)}$$
   then for any $m\geq 1$
$$\|f\|_{2m}\geq\eta\,|A|^{1/2m}\,|\Delta|\,.$$

   (b) Use Rudin's inequality (Question 8(c) on Examples Sheet 3) to deduce the strong Chang's dimension inequality: that $\Delta_\eta(A)$ is contained in $\mathrm{Span}(\Gamma)$ for some multiset Γ of size $O(\eta^{-2}\log(2/\alpha))$. [*Hint: Consider a maximal dissociated subset of* $\Delta_\eta(A)$.]

6. This question demonstrates how Bogolyubov-Ruzsa results were obtained in the days before almost-periodicity.

   (a) Show that if $E(A)\geq\delta|A|^3$ then, if
$$\Delta=\{\gamma\in\widehat{G}:|\widehat{1_A}(\gamma)|\geq\tfrac{1}{2}\delta^{1/2}|A|\},$$
   and B is the Bohr set with frequency set Δ and width $1/2$ then $B\subset 2A-2A$. [*Hint: First note that if* $x\in B$ *then* $\mathrm{Re}(\gamma(x))>1/2$ *for all* $\gamma\in\Delta$. *Then take real parts of the Fourier representation of* $1_A*1_A*1_{-A}*1_{-A}(x)$.]

   (b) Use part (a) with together with the strong Chang bound of Question 5 to deduce that if $A\subset\mathbb{Z}/N\mathbb{Z}$ with $|A|\geq N/K$ then $2A-2A$ contains a Bohr set with rank $O(K\log K)$ and width $1/K\log K$. Compare this to the Bogolyubov-Ruzsa lemma obtained in lectures.

7. Show that if $f:\{1,\ldots,N\}\to\mathbb{Z}$ has at least $K^{-1}N^3$ many $x,y,z,w\in\{1,\ldots,N\}$ such that
$$x+y=z+w\quad\text{and}\quad f(x)+f(y)=f(z)+f(w)$$
then there exists $a,b\in\mathbb{Q}$ such that
$$\#\{1\leq x\leq N:f(x)=ax+b\}\gg_K N^c,$$
where $c\gg_K 1$ is some constant depending only on K.

[*Hint: Consider the graph* $\Gamma=\{(x,f(x)):1\leq x\leq N\}\subset\mathbb{Z}^2$ *and use Question 3.*]

8. (a) Let $K\geq 1$. Show that if $A\subset\mathbb{Z}$ has $|A+A|\leq K\,|A|$ and $|A|$ is sufficiently large depending on K then A contains a non-trivial three-term arithmetic progression.

   (b) Explore what quantitative control you can get on how large is 'sufficiently large' using the bounds proved in lectures for Bourgain's theorem on three-term arithmetic progressions and the Freiman-Ruzsa-Sanders inverse theorem.