Additive Combinatorics Sheet 3

Lent Term 2021

Instructions

- These exercises are concerned with the material developed in Chapter 3: Almost-periodicity and applications.
- There are 9 exercises, of varying (and non-monotone) difficulty and length. **You are not expected to do them all**, but I have provided 9 for the enthusiast. If you have solved any 4 then this should be sufficient evidence (for yourself) that are you where you should be.
- The examples class will be run by Aled Walker, who will mark before the class your solutions to 2 exercises.
- The two exercises to be marked should be submitted by **9am Monday 26th April**. The class is **3:30pm Wednesday 28th April**. You should only submit the two solutions to be marked.
- Dr. Walker would appreciate knowing in advance of the class which exercises you found the most challenging. To this end, please submit the Self-Assessment form on Moodle before the class (even if you have not submitted any work to be marked).

1. This question indicates how (weaker) almost-periodicity results can be proved using only Fourier analysis. Let G be a finite abelian group of order N and $A \subset G$ with density $\alpha = |A|/N$.

   (a) For any $0 \leq \eta \leq 1$ let
   $$\Delta_\eta(A) = \left\{ \gamma \in \widehat{G} : |\widehat{1_A}(\gamma)| \geq \eta |A| \right\}.$$
   Use Parseval's identity to show that $|\Delta_\eta(A)| \leq \eta^{-2}\alpha^{-1}$.

   (b) Show that the set of L^2-almost periods of $1_A * 1_A$ with error $\epsilon |A|^{3/2}$ contains a Bohr set of rank $O(\epsilon^{-2}\alpha^{-1})$ and width $\gg \epsilon\alpha^{1/2}$. [*Hint: Write* $1_A * 1_A(x) = \mathbb{E}_\gamma \widehat{1_A}(\gamma)^2\gamma(x)$ *and consider separately the contribution from* $\gamma \in \Delta_\eta(A)$ *and* $\gamma \notin \Delta_\eta(A)$ *for a suitable choice of* η.]

   (c) Generalise your solution to part (b) to show that for any $m \geq 1$ the set of L^{2m}-almost periods of $1_A * 1_A$ with error $\epsilon |A|^{1+1/2m}$ contains a Bohr set of rank $O(\epsilon^{-2m}\alpha^{-1})$ and width $\gg \epsilon\alpha^{1/2m}$.

   (d) How does this compare to the Bohr set found by Theorem 10, the almost-periodicity result proved in lectures?

2. Show that if A, B, C are sets with $|C| \geq |B|$ such that there is S with $|A + S| \leq K|A|$ then the set T of L^∞-almost periods for $1_A * 1_B * 1_C$ with error $\epsilon |A| |B|$ has size
$$|T| \gg \exp(-O(\epsilon^{-2}(1 + \log(\tfrac{|C|}{|B|})) \log K)) |S|.$$

3. Suppose that A and x are such that $1_A * 1_A * 1_A(x) \geq \delta |A|^2$. Suppose further that there is some S such that $|A + S| \leq K|A|$. Show that, for any $k \geq 1$, there is a symmetric set X such that
$$x + kX \subset A + A + A$$
and
$$|X| \geq \exp(-O(k^2\delta^{-2}\log K)) |S|.$$

4. Show that if $A, B \subset \mathbb{F}_p^n$ with densities α, β respectively then $A+B$ contains a coset of a subspace of dimension $\gg \alpha\beta n/\log p$.

5. Let G be a finite abelian group and $A \subset G$ have density $\alpha = |A|/|G|$. Let $\epsilon > 0$ and T be the set of L^2-almost periods of $1_A * 1_A$ with error $\epsilon |A|^{3/2}$.

   (a) Show that if $\epsilon \geq 2$ then $T = G$.

   (b) Show that $|T| \gg \alpha^{O(\epsilon^{-2})} N$.

   (c) Show that T contains a Bohr set of rank $O(\epsilon^{-2}\alpha^{-1})$ and width $\gg \epsilon\alpha^{1/2}$.

   (d) Show that if $\Delta_\eta(A)$ is defined as in Question 1 then, for any $\eta > 0$, the set of almost-periods T is contained inside the Bohr set with frequency set $\Delta_\eta(A)$ and width $\eta^{-2}\epsilon\alpha^{-1/2}$.

6. (a) Show that if $X \subset \mathbb{F}_3^n$ is a symmetric set (so $X = -X$) such that $0 \in X$ and which contains at least k elements which are linearly independent over $\mathbb{F}_3$ then kX contains a subspace of dimension k.

   (b) Show that if $K \geq 4$ and $A \subset \mathbb{F}_3^n$ satisfies $|A+A| \leq K|A|$ then $A+A-A-A$ contains a subspace of dimension $\gg \sqrt{\log |A|}/\log K$.

7. Let $f : G \to \mathbb{C}$. In Lemma 26 we found by random sampling a function g which was the linear combination of $O(m\epsilon^{-2})$ characters such that $\|f-g\|_{2m} \leq \epsilon \|\widehat{f}\|_1 N^{1/2m}$. In this exercise we provide an example that shows that the linear dependence on m is necessary.

   Let $2m \leq n$ and choose some linearly independent $\gamma_1, \ldots, \gamma_{2m} \in \mathbb{F}_2^n$. We write $N = 2^n$ for the size of the group as usual. Let $f(x) = \frac{1}{2m}(\gamma_1(x) + \cdots + \gamma_{2m}(x))$. Out of all those functions which are the linear combination of $\leq m$ characters, let g be such that $\|f-g\|_{2m}$ is minimal.

   (a) Show that without loss of generality, the characters in g are from the subspace spanned by $\gamma_1, \ldots, \gamma_{2m}$. [*Hint: Consider what happens if we replace g by $g * 1_W$ where $W \leq \mathbb{F}_2^n$ is the subspace on which every γ_i is trivial.*]

   (b) Show that $N^{1/2m}|1-g(0)| \leq 2\|f-g\|_{2m}$. [*Hint: Consider what happens on the subspace of $\mathbb{F}_2^n$ orthogonal to $\gamma_1, \ldots, \gamma_{2m}$.*]

   (c) Let $V \leq \mathbb{F}_2^n$ be the subspace orthogonal to those characters in the definition of g. Show that

   $$\left(\sum_{x \in V} |f(x) - g(x)|^{2m}\right)^{1/2m} \geq \left(\frac{1}{2} - |1-g(0)|\right)|V|^{1/2m}.$$

   [*Hint: Use the triangle inequality and consider $1-f$ and $1-g$.*]

   (d) Deduce that, for any g which is the linear combination of at most m characters, we have

   $$\|f-g\|_{2m} \geq \tfrac{1}{8}\|\widehat{f}\|_1 N^{1/2m}.$$

   (And so in particular to approximate f better in L^{2m} norm we need $> m$ characters.)

8. We say that a set $D \subset G$ is *dissociated* if all $3^{|D|}$ sums of the form $\sum_{x \in D} c_x x$ where $c_x \in \{-1, 0, 1\}$ are distinct. Fix some dissociated set D.

   (a) Show that for any $\epsilon \in \{-1,1\}^D$, if

   $$P_\epsilon(\gamma) = \prod_{x \in D}(1 + \mathrm{Re}(\epsilon_x \gamma(x)))$$

   then $\mathbb{E}_\gamma |P_\epsilon(\gamma)| = 1$.

   (b) Show that for any $f : D \to \mathbb{C}$ and $\epsilon \in \{-1,1\}^D$, if

   $$F_\epsilon(\gamma) = \sum_{x \in D} \epsilon_x f(x)\overline{\gamma(x)},$$

   then $\widehat{f} = 2F_\epsilon * P_\epsilon$.

(c) Combining parts (a) and (b) with the ideas in the proof of the Marcinkiewicz-Zygmund inequality, prove Rudin's inequality: if D is dissociated then for any $f : D \to \mathbb{C}$ and any $m \geq 1$,

$$\|\widehat{f}\|_{2m} \ll m^{1/2} \|f\|_2 .$$

9. Let the *dimension* of a set A, denoted by $\dim(A)$, be the size of the largest dissociated subset of A.

(a) Show that if X is a dissociated set then for all $k \geq 1$ we have $|kX| \geq (Ck)^{-k} |X|^k$ for some absolute constant $C > 0$. [*Hint: Use Rudin's inequality from Question 7.*]

(b) Deduce that if $|A + A| \leq K |A|$ then $\dim(A) \ll K \log |A|$.

(c) Check to your satisfaction that the set of almost-periods produced by the proof of Theorem 12 can in fact be taken to be a subset of some fixed translate of S.

(d) Use the above ideas combined with Theorem 13 from lectures to show that if $|A + A| \leq K|A|$ there exists a set $X \subset A$ of size

$$|X| \geq \exp(-O((\log K)^4)) |A|$$

such that $\dim(X) \ll \log |A|$.