Additive Combinatorics Sheet 2

Lent Term 2021

Instructions

- These exercises use the material developed in Chapter 2: Fourier analysis, Bohr sets, and the density increment strategy.
- There are 11 exercises, of varying (and non-monotone) difficulty and length. **You are not expected to do them all**, but I have provided 10 for the enthusiast. If you have solved any 5 then this should be sufficient evidence (for yourself) that are you where you should be.
- The examples class will be run by Aled Walker, who will mark before the class your solutions to 2 exercises.
- The two exercises to be marked should be submitted by **9am Monday 8th March**. The class is **3:30pm Wednesday 10th March**. You should only submit the two solutions to be marked.
- Dr. Walker would appreciate knowing in advance of the class which exercises you found the most challenging. To this end, please submit the Self-Assessment form on Moodle before the class (even if you have not submitted any work to be marked).

1. Prove the Fourier inversion formula, that for any $f : G \to \mathbb{C}$,

$$f(x) = \mathop{\mathbb{E}}_{\gamma} \widehat{f}(\gamma)\gamma(x),$$

   both directly using orthogonality and also as a corollary of Parseval's theorem.

2. (a) Show that $|\widehat{1_A}(\gamma)| \in \{0, |A|\}$ for all $\gamma \in \widehat{G}$ if and only if A is a coset of subgroup.

   (b) Use Fourier analysis to prove that $|A + A| = |A|$ if and only if A is a translate of a subgroup. *(We have already seen a simple elementary proof of this fact, but it is instructive to find a Fourier analytic proof.)*

3. (a) Prove that if $A \subset G$ with density $\alpha > 0$ and $|\widehat{1_A}(\gamma)| \leq \delta |A|$ for all $\gamma \neq \mathbf{1}$ then, for any $x \in G$ and $k \geq 2$, we have

$$\left| 1_A * \cdots * 1_A(x) - \alpha |A|^{k-1} \right| \leq \delta^{k-2} |A|^{k-1},$$

   where the convolution is taken with k copies of A.

   (b) Deduce that if $k \geq 3$ and $|\widehat{1_A}(\gamma)| < \alpha^{1/(k-2)} |A|$ for all $\gamma \neq \mathbf{1}$ then $kA = G$.

4. The higher additive energies are defined, for any $m \geq 1$ and finite set A, by

$$E_{2m}(A) = \#\{(a_1, \ldots, a_{2m}) : a_i \in A \text{ and } a_1 + \cdots + a_m = a_{m+1} + \cdots + a_{2m}\}$$

   (so that e.g. the usual additive energy $E(A)$ is $E_4(A)$).

   (a) Show that

$$E_{2m}(A) = \mathop{\mathbb{E}}_{\gamma} \left| \widehat{1_A}(\gamma) \right|^{2m}.$$

(b) Use Hölder's inequality to show that if $n \geq m \geq 1$ (and $n > 1$) then

$$E_{2m}(A) \leq |A|^{\frac{n-m}{n-1}} E_{2n}(A)^{\frac{m-1}{n-1}}.$$

(c) Deduce that if $|A+A| \leq K|A|$ then for all $m \geq 2$ we have $E_{2m}(A) \geq K^{1-m}|A|^{2m-1}$. Compare this to what would follow by an application of Plünnecke's inequality.

5. (a) Using the density increment strategy and question 3, show that, for any $k \geq 3$, if $A \subset \mathbb{F}_p^n$ with density $\alpha = |A|/p^n$ then kA contains a coset of a subspace with codimension $O_k(\alpha^{-1/(k-2)})$.

   (b) If $|\mathbf{x}|$ is the Hamming weight of $\mathbf{x} \in \mathbb{F}_2^n$, i.e. the number of 1s in $\mathbf{x}$, then let

$$A = \{\mathbf{x} \in \mathbb{F}_2^n : |\mathbf{x}| \geq n/2 + \sqrt{n}\}.$$

   Show that (for large n) we have $|A| \geq \frac{1}{4}2^n$ and any coset of a subspace contained inside $A+A$ has codimension $\gg \sqrt{n}$. (In particular, in contrast to the situation for $k \geq 3$ in part (a), for $k = 2$ it is not possible to guarantee a coset of a subspace with codimension $O_\alpha(1)$ in $A+A$.)

6. Let $B = \mathrm{Bohr}(\Gamma; \rho)$ be a Bohr set of rank d and width $\rho \leq 1/2$ inside a finite abelian group G of order N.

   (a) Show that if $\gamma \in \Gamma$ then $|\widehat{1_B}(\gamma)| \geq \frac{1}{2}|B|$.

   (b) Deduce that $|B| \leq \frac{4}{d}N$.

   (c) Show that, for arbitrarily large $d \geq 1$ and $N \geq 1$, there exists a Bohr set B of rank d and width $\rho \leq 1/2$ in $\mathbb{Z}/N\mathbb{Z}$ such that $|B| \gg \frac{1}{d}N$.

7. Show that for any Bohr set B of rank d and dilate $0 < \delta < 1/2$, the Bohr set B is $(1/\delta)^{O(d)}$-covered by its dilate B_δ.

8. Let B be a regular Bohr set of rank d and width $\rho \leq 1/2$.

   (a) Show that for any $\eta, \delta > 0$ if $x \in B_\delta$ and $|\widehat{1_B}(\gamma)| \geq \eta|B|$ then

$$|1 - \gamma(x)| \ll \frac{\delta}{\eta}d.$$

   Hint: Consider the difference $\langle 1_B, \gamma\rangle - \langle 1_{B-x}, \gamma\rangle$.

   (b) Deduce that if $\Delta = \{\gamma : |\widehat{1_B}(\gamma)| \geq \frac{1}{2}|B|\}$ then there is an absolute constant $c > 0$ such that

$$B_{c\rho/d} \subset \mathrm{Bohr}(\Delta; \rho) \subset B.$$

9. Suppose we knew the following for some functions $D, \delta : [0,1] \to \mathbb{R}$:

> If $A \subset \mathbb{F}_p^n$ is a subset of density α that contains no non-trivial three-term arithmetic progressions than either
>
> (a) $|A| \ll p^{n/2}$ or
>
> (b) there is a subspace $V \leq \mathbb{F}_p^n$ of codimension $\leq D(\alpha)$ and a translate x such that $|(A-x) \cap V|/|V| \geq (1 + \delta(\alpha))\alpha$.

   (So that e.g. Lemma 15 in lectures is this with $D(\alpha) = 1$ and $\delta(\alpha) = \alpha/4$.)

   What upper bounds can you deduce for the maximal size of a subset of $\mathbb{F}_p^n$ which has no non-trivial three-term arithmetic progressions if...

   (a) $D(\alpha) \ll \alpha^{-1/2}$ and $\delta(\alpha) \gg \alpha^{1/2}$,

(b) $D(\alpha) \ll \alpha^{-1}$ and $\delta(\alpha) \gg 1$, or

(c) $D(\alpha) \ll 1$ and $\delta(\alpha) \gg 1$.

10. Suppose we knew the following for some functions $D, \delta : [0,1] \to \mathbb{R}$, for any finite abelian group G of odd order:

> Let B be a regular Bohr set of rank d and width ρ. If $A \subset B$ is a subset of density α that contains no non-trivial three-term arithmetic progressions than either
>
> (a) $|A| \ll (d/\alpha)^{O(d)} |B|^{1/2}$ or
>
> (b) there is a regular Bohr set $B' \subset B$ of rank $\leq d + D(\alpha)$ and width $\gg \rho(\alpha/d)^{O(1)}$ and a translate x such that $|(A-x) \cap B'| / |B'| \geq (1+\delta(\alpha))\alpha$.

(So that e.g. Lemma 20 in lectures is this with $D(\alpha) = 1$ and $\delta(\alpha) \gg \alpha$.)

What upper bounds can you deduce for the maximal size of a subset of $\{1, \ldots, N\}$ which has no non-trivial three-term arithmetic progressions if...

(a) $D(\alpha) \ll \alpha^{-1/2}$ and $\delta(\alpha) \gg \alpha^{1/2}$,

(b) $D(\alpha) \ll \alpha^{-1}$ and $\delta(\alpha) \gg 1$, or

(c) $D(\alpha) \ll 1$ and $\delta(\alpha) \gg 1$.

11. Let $m \geq 2$ and $p > m$ be prime.

(a) Sketch a proof that if $A \subset \mathbb{F}_p^n$ is a set of density α which has no solutions to $x_1 + \cdots + x_m = my$ with $x_1, \ldots, x_m, y \in A$ all distinct, then there is some constant $c_m > 0$ depending only on m such that either

i. $|A| \gg (p^n)^{1-c_m}$, or

ii. there is a subspace $V \leq \mathbb{F}_p^n$ of codimension 1 and a translate x such that

$$|(A-x) \cap V| / |V| \geq (1 + c_m \alpha^{\frac{1}{m-1}})\alpha.$$

(b) Deduce that if $A \subset \mathbb{F}_p^n$ contains no solutions to $x_1 + \cdots + x_m = my_m$ with all variables distinct then

$$|A| \ll_{p,m} \frac{p^n}{n^{m-1}}.$$

The energetic amongst you might also want to consider the same problem over $\mathbb{Z}/N\mathbb{Z}$.