Additive Combinatorics Sheet 1

Lent Term 2021

Instructions

- These exercises use the material developed in Chapter 1: sumset inequalities, the Balog-Szemerédi-Gowers lemma, Plünnecke's inequality, and covering results. The aim is to give you practice in working with these elementary concepts and increase your familiarity.
- There are 9 exercises, of varying (and non-monotone) difficulty and length. **You are not expected to do them all**, but I have provided 9 for the enthusiast. If you have solved any 4 then this should be sufficient evidence (for yourself) that are you where you should be.
- The examples class will be run by Aled Walker, who will mark before the class your solutions to 2 exercises, but you should submit your solutions to all the exercises.
- Work to be marked should be submitted by **9am Tuesday 16th February**. The class is **3:30pm Wednesday 17th February**.
- Dr. Walker would appreciate knowing in advance of the class which exercises you found the most challenging. To this end, please submit the Self-Assessment form on Moodle before the class (even if you have not submitted any work to be marked).

1. (a) Use Ruzsa's triangle inequality to show that if $|A+A| \leq K|A|$ then
$$|A-A| \leq K^2 |A|.$$
(b) Use Plünnecke's inequality to show that if $|A-A| \leq K|A|$ then
$$|A+A| \leq K^2 |A|.$$
(c) Show that for any set A
$$|A-A|^{3/4} \leq |A+A| \leq |A-A|^{4/3}.$$
[Hint: Use parts a) and b). These work well if we have good control on K. What bounds do you know that work even if we don't have control on K? Can you combine them?]
(d) Show that the exponent of 2 in part (a) is best possible by considering
$$A = \left\{ (x_1, \ldots, x_d) \in \mathbb{N}^d : \sum x_i \leq n \right\}$$
for large n and d. (Don't worry about rigorously calculating/showing this, but try to roughly see why this example works.)

2. Recall that if $A \subset \mathbb{Z}$ satisfies $|A+A| = 2|A| - 1$ then A must be an arithmetic progression. In this exercise we will prove Vosper's theorem, which is the analogous result in $\mathbb{F}_p$.
Let $A, B \subset \mathbb{F}_p$ be sets such that $|A|, |B| \geq 2$ and $|A+B| = |A| + |B| - 1 \leq p - 2$.
(a) Show that if either A or B is an arithmetic progression then the other must be an arithmetic progression with the same step.

(b) Show that if $A+B$ is an arithmetic progression then A and B must both also be arithmetic progressions of the same step. *[Hint: Consider $C-B$ where C is the complement of $A+B$.]*

(c) Using the previous two parts and induction on $|B|$, prove that in fact A and B are always arithmetic progressions of the same step. *[Hint: What happens in our proof of the Cauchy-Davenport lemma if equality holds?]*

3. Show that if $|A+B| \leq K|A|$ then for any $\epsilon > 0$ there is $X \subset A$ such that $|X| \geq (1-\epsilon)|A|$ and

$$|X + kB| \leq \epsilon^{-k} K^k |X|.$$

[Hint: What does Plünnecke's inequality give you? What happens if you remove the X given by Plünnecke from A and apply the inequality again?]

4. In Plünnecke's inequality we go from a bound $|A+B| \leq K|A|$ to a bound on $|X+B+B|$ for some unspecified $X \subset A$. This exercise asks what kind of bounds we can expect for $|A+B+B|$ itself.

(a) Show that if $|A+B| \leq K|A|$ then there exists $A' \subsetneq A$ such that

$$|A+B+B| \leq |A'+B+B| + K^2(|A| - |A'|).$$

(b) Let B be a fixed set and $M \geq 1$ also be fixed. Show that for any $N \geq M/|B+B|^{1/2}$, for all A such that $|A| \leq N$ and $|A+B| \leq M$ we have

$$|A+B+B| \leq 3M|B+B|^{1/2} - \frac{M^2}{N}.$$

[Hint: Use induction on N.]

(c) Deduce that, for any sets A and B, if $|A+B| \leq K|A|$ then

$$|A+B+B| \ll K^2 |A|^{3/2}.$$

(d) Show that the previous bound is best possible, in that there exist arbitrarily large A and B with $|A+B| \ll |A|$ and $|A+B+B| \gg |A|^{3/2}$. *[Hint: You might want to look at finite subsets of $\mathbb{Z}^3$ formed by taking the union of truncated 'lines' and 'planes'.]*

5. Show that the following are equivalent 'up to polynomial losses', in that if one property holds with parameter K then the others hold with parameters $K^{O(1)}$:

(a) $|A+A| \leq K|A|$,

(b) there exists B such that $|A+B| \leq K|A|^{1/2}|B|^{1/2}$,

(c) there exists a symmetric set H containing the origin such that $H+H \subset H+X$ for some $|X| \leq K$, and $A \subset x+H$ for some x, and $|A| \geq K^{-1}|H|$.

(Such an H is called a *K-approximate group*, and understanding the structure of these in non-abelian groups in particular is a vibrant area of current research. This exercises shows that understanding sets of small doubling is equivalent to understanding approximate groups.)

6. Show that if $|A+A-A-A| < 2|A|$ then $A-A$ is a group.

7. Adapt the proof of the Balog-Szemerédi-Gowers lemma to show to your satisfaction that the following asymmetric version holds: Let $E(A,B)$ count the number of solutions to $a_1+b_1 = a_2+b_2$ with $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Show that if $E(A,B) \geq K^{-1}|A|^{3/2}|B|^{3/2}$ then there exist $A' \subset A$ and $B' \subset B$ such that

(a) $|A'| \gg K^{-1}|A|$,

(b) $|B'| \gg K^{-1}|B|$, and

(c) $|A' - B'| \ll K^{O(1)} |A|^{1/2} |B|^{1/2}$.

8. (a) Prove the following generalisation of Plünnecke's inequality: if we have h sets $B_1, \ldots, B_h$ such that $|A + B_i| \leq K_i |A|$ for $1 \leq i \leq h$ then there is an $X \subset A$ such tht

$$|X + B_1 + \cdots + B_h| \ll_h K_1 \cdots K_h |X|$$

and in particular

$$|B_1 + \cdots + B_h| \ll_h K_1 \cdots K_h |A|.$$

[The easiest way is to use Plünnecke's inequality as a black box, and apply it to $B = \cup(B_i + T_i)$ where T_i are sets of some suitably chosen size such that the sums $y + t_1 + \cdots + t_h$ are disjoint for all $y \in A + B_1 + \cdots + B_h$ and $t_i \in T_i$.]

(b) By considering what happens if we replace A by $A \times \cdots \times A$ and B_i by $B_i \times \cdots \times B_i$ show that the second conclusion can be upgraded to

$$|B_1 + \cdots + B_h| \leq K_1 \cdots K_h |A|.$$

9. This exercise shows how to improve the exponent in the Balog-Szemerédi-Gowers lemma. This proof is due to Schoen (2014) and the exponent of K^3 here remains the best known - if you can do any better on the exponent by any method, that would be big news!

Fix some set A such that $E(A) \geq K^{-1} |A|^3$.

(a) Let $G \subset A^2$ be the set of pairs (a, b) such that $1_A \circ 1_A(a - b) < cE(A)/|A|^2$. Show that there exists some $\frac{1}{4K} \leq \lambda < 1$ such that if

$$S = \{x : \lambda |A| < 1_A \circ 1_A(x) \leq 2\lambda |A|\}$$

then

$$\sum_{(a,b) \in G} |(A - a) \cap (A - b) \cap S| \leq 8c\lambda^2 |S| |A|^2.$$

[Hint: Consider λ of the form 2^{-i} and sum the quantities on both sides over an appropriate range of i.]

(b) By considering X of the form $A \cap (A + s)$ where s is chosen uniformly at random from S, show that there is $X \subset A$ of size $|X| \gg K^{-1} |A|$ such that for all but at most $c |X|^2$ many pairs $(a, b) \in X^2$ we have

$$1_A \circ 1_A(a - b) \gg cK^{-1} |A|.$$

(c) Deduce that there exists $A' \subset A$ with $|A'| \gg K^{-1} |A|$ and $|A' - A'| \ll K^3 |A|$.